On the non-existence of elements of Kervaire invariant one

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Milnor’s Questions
How many smooth structures are there on the $n$-sphere?

Theorem (Poincaré Conjecture: Smale-Freedman-Perelman)

*If $M$ is a homotopy $n$-sphere that is a manifold, then $M$ is homeomorphic to $S^n$.***
Definition

Let $\Theta_n$ be the group of h-cobordism classes of homotopy $n$-spheres with addition connect sum.

Have a map

$$\Theta_n \xrightarrow{\psi_n} \pi_n^s / \text{Im}(J).$$
Pontryagin’s Work

Definition

A framed $n$-manifold is an $n$-manifold with a continuous choice of basis for the normal vectors at every point.

$$\{ M^n \subset \mathbb{R}^{n+k} \}/\text{cobordism} \quad \longrightarrow \quad \pi_{n+k} S^k$$

Framings on $S^k$ \quad \longrightarrow \quad \text{Im}(J)$
Pontryagin’s Computations

\[ \pi_0^S = \mathbb{Z} : \]

\[ \pi_1^S = \mathbb{Z}/2\mathbb{Z} : \]
Framed Surgery

$\pi_2^S$:

Pontryagin: **framed surgery**
Consequences

$\not\exists \text{ not onto}$

The map

$$\Theta_2 \to \pi_2^s/\text{Im}(J)$$

is not surjective.

Get a map

$$\mu : H_n(M; \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}.$$  

If we can do surgery: 0, if we can’t: 1.
Definition

Let $bP_{n+1}$ be the subset of $\Theta_n$ of those spheres that bound parallelizable (frameable) manifolds.

Theorem (Kervaire-Milnor)

If $n \not\equiv 2 \mod 4$, then there is an exact sequence

$$0 \rightarrow bP_{n+1} \rightarrow \Theta_n \xrightarrow{\psi_n} \pi_n^s/\text{Im}(J) \rightarrow 0.$$  

If $n \equiv 2 \mod 4$, then there is an exact sequence

$$0 \rightarrow bP_{n+1} \rightarrow \Theta_n \xrightarrow{\psi_n} \pi_n^s/\text{Im}(J) \xrightarrow{\Phi_n} \mathbb{Z}/2 \rightarrow bP_n \rightarrow 0.$$  

Hill Framed Manifolds & Equivariant Homotopy
$bP_{n+1}$ has a simple structure: it’s finite cyclic!

**Theorem (Kervaire-Milnor)**

$$|bP_{n+1}| = \begin{cases} 
1 & n \equiv 0 \mod 2 \\
1 \text{ or } 2 & n \equiv 1 \mod 4 \\
2^{2k-2}(2^{2k-1} - 1)\text{num}\left(\frac{4B_k}{k}\right) & n = 4k - 1 > 3.
\end{cases}$$

**Theorem (Adams, Mahowald)**

$$|\text{Im}(J)| = \begin{cases} 
1 & n \equiv 2, 4, 5, 6 \mod 8, \\
2 & n \equiv 0, 1 \mod 8, \\
\text{denom}\left(\frac{B_k}{4k}\right) & n = 4k - 1.
\end{cases}$$
**Definition (Kervaire Invariant)**

If $M$ is a framed $(4k + 2)$-manifold, then the Kervaire invariant $\Phi_{4k+2}$ is the obstruction to surgery in the middle dimension.

**Kervaire Invariant One Problem**

Is there a smooth $n$-manifold of Kervaire invariant one?
Adams Spectral Sequence

There is a spectral sequence with

\[ E_2 = \text{Ext}_A(H^*(Y), H^*(X)) \]

and converging to \([X, Y]\).

- (Adem) \( \text{Ext}^1(F_2, F_2) \) is generated by classes \( h_i, i \geq 0 \).
- \( h_j \) survives the Adams SS if \( \mathbb{R}^{2^j} \) admits a division algebra structure:
  \[ d_2(h_j) = h_0 h_{j-1}^2. \]
Browder’s Reformulation

Theorem (Browder 1969)

1. There are no smooth Kervaire invariant one manifolds in dimensions not of the form \(2^{j+1} - 2\).
2. There is such a manifold in dimension \(2^{j+1} - 2\) iff \(h_j^2\) survives the Adams spectral sequence.

Classical Examples

- \(h_1^2: S(\mathbb{C}) \times S(\mathbb{C})\)
- \(h_2^2: S(\mathbb{H}) \times S(\mathbb{H})\)
- \(h_3^2: S(\mathbb{O}) \times S(\mathbb{O})\)
Adams Spectral Sequence
Adams Spectral Sequence
Previous Progress

Theorem (Mahowald-Tangora)

The class $h_4^2$ survives the Adams SS.

Theorem (Barratt-Jones-Mahowald)

The class $h_5^2$ survives the Adams SS.
Main Theorem

Theorem (H.-Hopkins-Ravenel)

For \( j \geq 7 \), \( h^2_j \) does not survive the Adams SS.

We produce a cohomology theory \( \Omega^*(-) \) such that

1. the cohomology theory detects the Kervaire classes,
2. \( \Omega^{-2}(pt) = 0 \), and
3. \( \Omega^{k+256}(X) \cong \Omega^k(X) \).

We rigidify to a \( C_8 \)-equivariant spectrum \( \Omega_C \):

\[
\Omega = \Omega_C^{C_8} \cong \Omega^{hC_8}.
\]
Cohomology Theories

Cohomology Theory

\[
\{ \text{Topological Spaces} \} \xrightarrow{E^*} \{ \text{Graded Abelian Groups} \}
\]

satisfying

1. Homotopy Invariance: \( f \simeq g \Rightarrow E^*(f) = E^*(g) \)
2. Excision: \( X = B \cup_{A} C \), then have a long exact sequence

\[ \cdots \rightarrow E^n(X) \rightarrow E^n(C) \oplus E^n(B) \rightarrow E^n(A) \rightarrow E^{n+1}(X) \rightarrow \cdots \]

Example

1. Singular cohomology
2. \( K \)-theory (vector bundles on \( X \) )
Idea

Spectra represent cohomology theories: $E^n(X) = [X, E_n]$

Spectrum

A sequence of spaces $E_1, E_2, \ldots$ together with equivalences

$$E_n \cong \Omega E_{n+1} = \text{Maps}(S^1, E_{n+1})$$

1. Singular homology: $H\mathbb{Z}_n = K(\mathbb{Z}, n)$
2. $K$-Theory: $KU_{2n} = \mathbb{Z} \times BU, KU_{2n-1} = U.$
Equivariant Homotopy

Homotopy theory for spaces with a $G$-action.

- For $H \subset G$, have “fixed points” $E^H$.
- There are spheres for every real representation.

Example

If $G = \mathbb{Z}/2$, then we have $S^{\rho_2} = \mathbb{C}^+$ and $S^2$.

$$ (S^{\rho_2})^{\{e\}} = S^2 \quad (S^{\rho_2})^{C_2} = \mathbb{R}^+ = S^1. $$
What is our cohomology theory?

1. Begin with the bordism theory for (almost) complex manifolds: $MU$.

2. Localize: $\Omega_{\emptyset} = \bar{\Delta}^{-1} MU^{(C_8)}$.

Theorem

For a finite index subgroup $H \subset G$, there is a multiplicative functor

$$N^G_H : H-\text{Spectra} \to G-\text{Spectra}.$$
Goal

Want to compute the homotopy groups of the fixed points for spectra like $\Omega$. 

Start with Schubert cells: The Grassmanians $Gr_n(\mathbb{C}^k)$ all have cells of the form $\mathbb{C}^m$ 

For $MU^{(C_8)}$, we therefore see three kinds of representation spheres:

1. $S^{k\rho_8}$
2. $Ind_{C_4}^{C_8} S^{k\rho_4}$
3. $Ind_{C_2}^{C_8} S^{k\rho_2}$
Introduction

Slice Spectral Sequence

Slice Basics

Gap Theorem: \( \pi_{-2} \Omega = 0 \)

HFP & Periodicity: \( \pi_{k+256} \Omega = \pi_k \Omega \)

Advantages of the Slice SS

Hill Framed Manifolds & Equivariant Homotopy
Advantages of the Slice SS
Slice Filtration of $MU^{(C_8)}$

**Theorem**

There is a multiplicative filtration of $MU^{(C_8)}$ with associated graded

$$\bigvee_{p \in \mathcal{P}} \text{Ind}_{H(p)}^{C_8} S^{k(p)} \rho_{H(p)} \wedge H\mathbb{Z}.$$ 

**Corollary**

There is a spectral sequence

$$E_2^{s,t} = \bigoplus_{p \in \mathcal{P}, |p| = t} H_{t-s}^{H(p)} \left( S^{k(p)} \rho_{H(p)} + V ; \mathbb{Z} \right) \Longrightarrow \pi_{t-s} V MU^{(C_8)}.$$
Key Fact

The $E_2$-term can be computed from equivariant simple chain complexes.

Cellular Chains for $S^{ρ_4-1}$

$$
\begin{align*}
C_\bullet(S^3) & \xleftarrow{\mathbb{Z}} \mathbb{Z}^2 & \xleftarrow{\mathbb{Z}^4} & \mathbb{Z}^4 \\
C_\bullet(S^1) &
\end{align*}
$$
### Gaps

**Theorem**

For any non-trivial subgroup $H$ of $C_8$ and for any induced sphere $\text{Ind}_{K}^{C_8} S^{k\rho_K}$ with $k \in \mathbb{Z}$,

$$H^K_{-2} (S^{k\rho_K}; \mathbb{Z}) = 0$$

1. If $k \geq 0$ or $k < -2$, then $C_{-2} = 0$
2. If $k = -1, -2$, in the relevant degrees, the complex is $\mathbb{Z} \to \mathbb{Z}^2$ by $1 \mapsto (1, 1)$.

**Corollary**

For any $k$, $\pi_{-2} \Sigma^{k\rho_8} MU(C_8) = 0$. 

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Slice Spectral Sequence

Slice Basics

Gap Theorem: $\pi_{-2}\Omega = 0$

HFP & Periodicity: $\pi_{k+256}\Omega = \pi_k\Omega$
Euler and Orientation Classes

Homology of representation spheres is generated by Euler classes and orientation classes for representations.

Theorem

1. The fixed and homotopy fixed points of $\Omega$ agree if all Euler classes are nilpotent.
2. The homotopy of the homotopy fixed points of $\Omega$ is periodic if some regular representation is orientable.