Computations towards $K(ko)$

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Outline

1. Motivation
   - The Goal: Chromatic Red Shift
   - Previous Work

2. Our Results
   - Main Results
   - Ideas and Goals
Basic Setup

- \(K(-)\) is a functor from commutative rings / commutative \(S\)-algebras to commutative \(S\)-algebras.
- \(K(-)\) is very hard to compute but has wide-ranging applications.
- Basic step: Approximate \(K(A)\) by \(\text{THH}(A) = \text{Tor}^{A \otimes A^{\text{op}}}(A, A)\).
From \( \text{THH} \) to \( K \)

- \( \text{THH}(A) \) has a natural action of \( S^1 \)
- There is a natural map \( K(A) \to \text{THH}(A) \) that factors through the \( S^1 \) fixed points.
- At \( p \), the \( S^1 \) fixed points are essentially \( TC(A) \).
- For connective \( A \), \( K(A) \to TC(A) \) is almost an equivalence.
Conjecture

If $A$ is a ring spectrum of type $n$, then $K(A)$ has type $n + 1$.

Examples of type $n$ ring spectra:

- $H\mathbb{Q}$ (n=0)
- $\ell$ (n=1)
- $BP\langle n \rangle$. 
Bökstedt / McClure-Staffeldt: Computed $H_* THH(\ell)$, effectively also giving the $V(1)$-homology:

$$V(1)_* THH(\ell) = E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p[\mu_2].$$

McClure & Staffeldt: Computed the $V(0)$-homology:
Everything but $\lambda_1$ is $v_1$-torsion.

Angeltveit & Rognes: Extended these results to all primes.
The $V(1)$ Homology of $K(\ell)$

- Ausoni & Rognes: Computed the $V(1)$ homotopy of $K(\ell)$ for $p > 3$ in several steps:
  - Calculate $V(1)_* THH(\ell)^{C_p^n}$ and the Tate analogue.
  - Use this to calculate $V(1)_* TC(\ell)$.
  - Used the trace map $V(1)_* K(\ell) \to V(1)_* TC(\ell)$ to understand $V(1)_* K(\ell)$.
- Extended by Ausoni to $p = 3$ and to $ku$. 
THH*(ℓ) and THH*(ko)

Theorem

\[ p\text{-locally, } \overline{\text{THH}}_*(\ell) = F \oplus T \]

- \( F \) is \( p \)-torsion free and \( v_1 \) becomes increasingly \( p \) divisible.
- \( T \) is \( p \) and \( v_1 \)-torsion and is a sum of self-dual modules.

Theorem

\[ 0 \to F_{\text{ko}} \to \overline{\text{THH}}_*(\text{ko}) \to T_{\text{ko}} \to 0 \]

- \( F_{\text{ko}} \) is \( \eta^2 \)-torsion and \( v_1^4 \) becomes increasingly 2 divisible.
- \( T_{\text{ko}} \) is 2, \( \eta \), and \( v_1^4 \) torsion and is a sum of self-dual modules.
A Picture of 2-torsion in $\text{THH}_*(ku)$
A Picture of $THH_*(ko)$
• $THH(-)$ has a relative version $THH(A; M)$.
• $\ell$ has 4 natural bimodules: $\ell, \ell/p, H\mathbb{Z}(p)$, and $H\mathbb{F}_p$.
• These give Bockstein spectral sequences for adding in $p$ or $v_1$.
• The end results must be the same.
• $ko$ has one extra bimodule: $ku$. 
There is an associative ring spectrum $Y_2$ such that

$$H_\ast(Y_2) = \mathbb{F}_p[\xi_1, \xi_2] \otimes E(\tau_0, \tau_1).$$

$$Y_{2\ast}(THH(\ell)) = \mathbb{F}_p[\xi_1, \xi_2, \mu_2] \otimes E(\lambda_1, \lambda_2)$$

For big enough values of $p$, $Y_2$ is filtered by $V(1) \Rightarrow$ Can play a similar game to understand homotopy and Tate fixed points.
Future Work

- Replace “magical” and subtle theorem used by Rognes and Ausoni to understand all homotopy fixed points.
- Compute $Y_2^*(THH(\ell)^{hS^1})$ and $Y_2^*TC(\ell)$.
- Run the $Y_2$-based Adams spectral sequence to get $TC_*(\ell)$. 
Can understand $THH(\ell)$ and $THH(ko)$.

Can use a special spectra $Y_2$ to try to compute $TC(-)$. 