1 Background and Outline

Central to algebraic topology are computations and the techniques to evaluate and interpret them. These radiate out into modern mathematics, answering questions in geometric topology, algebraic geometry, and more. They also provide a framework for further analysis and lead to refinements of previously understood work.

From its inception in Poincaré's *Analysis Situs*, algebraic topology has played algebraic data and techniques against geometric ones. A spectacular example is Pontryagin's seminar work on framed bordism, in which he showed framed manifolds up to bordism are the same as homotopy classes of maps between spheres. He developed the techniques of surgery theory to compute these, successfully determining the first two stable stems geometrically. He failed in the next dimension only to recognize that the obstruction to framed surgery, known now as the Kervaire invariant, need not vanish in dimension 2.

A great deal of work in algebraic topology since has revolved around settling the Kervaire invariant one problem: when are framed manifolds not frame bordant to spheres. Kervaire described a generalization of Pontryagin's work [34], using it to show the existence of a non-smoothable 10-manifold. Kervaire-Milnor built on this and Milnor's exotic 7-spheres to link the number of exotic smooth structures on the n-sphere to the n^{th} stable homotopy group of spheres [42], [35], showing that the subgroup of the stable stems corresponding to framed homotopy spheres has index at most two in the stable stems and that the quotient is generated by a manifold of Kervaire invariant one. This was distilled by Browder to a question of the survival of a particular family of elements in the Adams spectral sequence [11].

Hopkins, Ravenel, and I recently solved this problem, using techniques and computations in C_8-equivariant homotopy theory [29]. There is a rich and complicated literature on equivariant homotopy theory for finite and compact Lie groups G. However computations are very difficult and few are known. The heart of the projects described below is to use the skills, techniques, and understanding garnered from the solution to broaden our knowledge of equivariant computations and, by universality of certain maps, of the homotopy groups of spheres.

Equivariant methods have a rich life in the homotopical approaches to algebraic K-theory. Bökstedt showed that there is a genuine S^1-equivariant model for topological Hochschild homology (THH) [10], and the Dennis trace from the algebraic K-groups to the Hochschild homology refines to a map of spectra from the K-theory spectrum to THH. Building on this, Bökstedt-Hsiang-Madsen and Hesselholt-Madsen produced a tower of spectra built out of the fixed points of THH for finite subgroups of S^1 [8], [24]. The trace map lifts to the limit of this tower and serves as a good approximation to the algebraic K-theory. These methods have produced beautiful results, showing close connections between the equivariance and algebraic geometry, and they suggest an interplay between K-theory and chromatic height, dubbed "chromatic red-shift" by Rognes. Projects below use tools from Hill-Hopkins-Ravenel to produce new models of THH with nice formal properties and to push further computations.

Kervaire Invariant One Problem and new equivariant tools

Building on our previous collaborative efforts to understand the homotopy groups of the Hopkins-Miller spectra (a continuation of which is described in Section 3.2), Hopkins, Ravenel, and I proved the following theorem [29].

Theorem 1. *There are manifolds of Kervaire invariant one only in dimensions* 2*,* 6*,* 14*,* 30*,* 62*, and possibly* 126*.*

To prove this, we lifted Browder's result to the Adams-Novikov spectral sequence and then recast it first as a question about the C_8-homotopy fixed points of a particular test spectrum $\Omega_{\mathbb{O}}$ (which is a convenient stand-in for the Hopkins-Miller spectrum $EO_4(C_8)$). This we then rephrased as a question in genuine equivariant homotopy theory, showing

1. the homotopy group $\pi_{-2}\Omega_{\mathbb{O}}^{C_8}$ is zero,
2. the homotopy groups of $\Omega_{\mathbb{O}}^{hC_8}$ are 256 periodic, and
3. the natural map $\Omega_{\mathbb{O}}^{C_8} \to \Omega_{\mathbb{O}}^{hC_8}$ is an equivalence.

Proving these three steps required new tools and approaches, all of which continue to inform my research and all of the proposed projects. Of these, the most important are the norm functor (which is used to build $\Omega_{\mathbb{O}}$ as a C_8 commutative ring spectrum) and the slice filtration (elementary computations in the associated spectral sequence show the remaining results).

The norm

Constructing $\Omega_{\mathbb{O}}$ required a way to multiplicatively induce an H-spectrum to a G-spectrum. This is achieved by our norm functor. We begin with the Landweber-Araki Real bordism spectrum $MU_{\mathbb{R}}$ and apply the norm to get a G-spectrum $MU^{(G)}$, the underlying spectrum for which is just $MU^{\wedge(|G|/2)}$. Then $\Omega_{\mathbb{O}}$ is a localization of $MU^{(C_8)}$.

The norm has an algebraic antecedent, the Evens transfer [18], and homotopical versions were studied by Greenlees-May (where we see that the norm gives rise to equivariant power operations) [22]. The underlying concept is simple: take an H-spectrum X and smash together G/H (twisted) copies of X. Thus if X is a representation sphere S^V, then $N_H^G S^V = S^{\mathrm{Ind}\, V}$.

Since the smash product is the coproduct in commutative ring spectra, it comes as no surprise that the norm is the left adjoint to the forgetful functor from G commutative ring spectra to H commutative ring spectra. The properties of a left adjoint were instrumental in producing G-equivariant families out of H-equivariant maps from spheres, leading, for example, to a method of killing a G-equivariant collection of homotopy elements.

Actually proving much of what we needed required a more general (and then conceptually cleaner) description. If we consider symmetric monoidal diagram categories, then we have norms associated to morphisms of indexing categories analogous to covering spaces. For equivariant spectra, this gives both classical induction (using the symmetric monoidal structure induced by the wedge) and our norms (using the smash product), and it provides a way to interpret both as an enrichment of the symmetric monoidal product to G-sets. This extension allows a unified treatment of the constructions, basic computations, and homotopical properties. It also provides a clean framework for the localization work described in Section 2 and the on-going work on THH described in Section 4.

The slice filtration

The slice filtration is an equivariant filtration generalizing work of Dugger for $G = C_2$ and loosely modeled on the motivic analogue due to Voevodsky [15], [50, 52, 51]. The C_2-equivariant version was studied by Hu-Kriz in their analysis of $MU_{\mathbb{R}}$ [32], and they provide a nice connection between it and the motivic version [33].

The construction mirrors the classical Postnikov tower. To form the stages of the Postnikov tower, we nullify all maps from spheres of some fixed connectivity. In the equivariant case, we kill the G-space of all maps from spheres of a fixed connectivity, which is equivalent to nullifying all induced spheres of that connectivity. For the slice tower, we use a new kind of connectivity (building on the larger number of spheres in the equivariant context). We declare that the spectra $G_+ \wedge_H S^{k\rho_H - \epsilon}$ are n-slice connected whenever $k|H| - \epsilon > n$ (here and henceforth, ρ_H denotes the regular representation of H, while ϵ is either 0 or 1). The slice tower is the associated nullification tower.

One important way that this differs from the Postnikov tower is that if k is non-negative, then $G_+ \wedge_H S^{k\rho_H - \epsilon}$, while slice $(k|H| - \epsilon - 1)$-connected, is only $(k - \epsilon - 1)$-connected as an equivariant spectrum. This means that nullification tower for slice n-connected things can change homotopy groups below dimension n, and the slice tower spreads each homotopy group over several layers.

For a general G and X, we know almost nothing about the slices of X or the associated slice spectral sequence. With the exception of slices in dimensions congruent to 0 or -1 modulo the order of G (which are equivalent to full subcategories of the category of Mackey functors), we know little about the categories of n-slices. The slice primer described in Section 3.1 explores this in more detail. Nevertheless, for $MU^{(G)}$, we determined all of the slices, showing that the odd slices are contractible and the even slices are wedges of spectra of the form $G_+ \wedge_H S^{k\rho_H} \wedge H\underline{\mathbb{Z}}$. The number of summands and the isotropy are all determined combinatorially by the structure of the underlying homotopy as a G-module.

It is immediate that the same result is true for $\Omega_{\mathbb{O}}$, from which we see the vanishing result already on the E_2-page of the slice spectral sequence. A basic understanding of the differentials gave the needed periodicity and showed that the fixed and homotopy fixed points agree. One of the projects described below is to complete these computations and apply them to Hopkins-Miller spectra.

Outline

In Sections 2 through 4, I will describe the major projects I intend to work on. Particular goals will be listed, both as "goals" and as "conjectures". Section 2 describes work with Hopkins on equivariant localizations and questions related thereto. Section 3 begins with a discussion of my slice primer and continues with problems related to determining the homotopy groups of the higher real K-theories and $MU^{(G)}$. Section 4 explores the impact of norm and slice machinery on computations in topological Hochschild homology.

I will discuss the broader impacts of the project in Section 5. The final section documents my prior support and the publications that resulted from it.

2 Equivariant Localizations and GL_1

While working to reconcile the natural connections between G/N-spectra and G spectra with the obvious symmetric monoidal structures, I discovered that localizations of equivariant commutative ring spectra need not be commutative ring spectra. This was wildly unsettling! However, it became clear that this should not be expected; even in the algebraic context of Green functors, arbitrary localizations do not preserve Tambara functors (the algebraic structure exhibited by equivariant commutative rings [12]). On the other hand, inverting norms from the trivial group does.

Hopkins and I built on this algebraic understanding, recognizing that the key issue is that an arbitrary localization need not preserve norms. With this insight, it was easy to see a sufficient condition which would allow copying the standard proof that localization preserves commutative rings found in Elmendorf-Kriz-Mandell-May [16].

Theorem 2. *If the norm of an acyclic spectrum is acyclic, then the localization preserves commutative rings.*

The cleanest argument for this actually works for enriched symmetric monoidal categories. Equivariant commutative ring spectra are tensored over G-sets (and therefore G-spaces), and this tensoring operation actually arises from an enriched tensor product on G-spectra. Hopkins and my result can now be reformulated in a more general way: if the category of acyclics is a symmetric monoidal category enriched over G-sets, then localization preserves commutative rings.

Along the way, we noticed that there is a hierarchy of notions of "commutative" in the equivariant context. These are the multiplicative analogue of the various notions of equivariant spectra, interpolating between naive and genuine and modeled on the representations by which we can desuspend. There is a straightforward, operadic definition of these forms of commutative algebras: algebras over the operad $\mathcal{L}(U) = \mathcal{L}(U^n, U)$, the linear isometries operad based on a G universe U. This is an operad in G-spaces, and all of the spaces in it are underlain by an E_∞ operad. The key fact is that a non-trivial G-action on $\mathcal{L}(U)$ can produce fixed points in the algebras over the operad. The two extreme examples are U the trivial universe, which gives us no new fixed points (so in particular the free algebra over this on G_+ has no geometric fixed points) and U a complete universe, the algebras over which are commutative algebras (and these have interesting fixed points).

These notions have an intuitive formulation using the enriched symmetric monoidal structure or equivalently using the norm. If G/H embeds in U and R is an $\mathcal{L}(U)$-algebra, then we have a map $N_H^G(R) \to R$. This approach then gives a refinement of the earlier stated result.

Theorem 3. *If R is a genuine G-commutative ring spectrum, and if N_H^G preserves acyclics for all H such that G/H embeds in U, then the localization of R is a $\mathcal{L}(U)$-algebra.*

In particular, since N_G^G is the identity, we see that at the very worst, localization takes G-commutative ring spectra to algebras over the trivial linear isometries operad. Building on this requires relatively straightforward result that must be known to experts.

Conjecture 2.1. *There are symmetric monoidal category structures on genuine equivariant spectra for which $\mathcal{L}(U)$-algebras are the commutative algebras, just as for S-modules.*

This is similar to Elmendorf-May's work on commutative rings in equivariant spectra indexed by a universe U [17]. They key difference is that here there is a different universe indexing the additive and multiplicative structures.

The localization result and the companion results about the norm interlace the various symmetric monoidal structures. One exciting application of this work is the different versions of GL_1 in the equivariant context. In the case of G a finite group (the only case we consider), GL_1 of an $\mathcal{L}(U)$-algebra is again an algebra over $\mathcal{L}(U)$, and these are simply equivariant spectra indexed on U rather than on a complete G-universe. The norm maps seem to give rise to the needed transfer maps on GL_1.

Goal 2.2. *Understand the interplay between the multiplicative structure with its norms and the additive structure of GL_1 with its transfers.*

3 Equivariant computations, $EO_n(G)$, and $MU^{(G)}$

3.1 Slice Filtration Primer

There are very few equivariant computations in the literature. Since the slice spectral sequence provides a fantastic tool for approaching these, I wrote a self-contained introduction [26]. It began with a review of the salient features from the Hill-Hopkins-Ravenel paper, and then it answered several questions posed to me at various talks on the slice filtration. In particular, I showed several criteria which allow a direct comparison of "slice dimension" and CW-dimension, building a large number of spaces and spectra for which the slice dimension is known. For example, if X is the k-skeleton of an n-dimensional CW-spectrum Y such that the slice dimension of Y is at least n, then the slice dimension of X is at least k. Similarly, if $V = \mathbb{R}\cdot X$ is the permutation representation associated to a G-set X, then the slice dimension of S^V is at least $|X|$.

Using these slice dimension results, I turned to Mackey functors and the Burnside ring. For the groups C_{p^n}, I showed that the topological slice filtration of the spectrum $H\underline{M}$ for a Mackey functor $\underline{M}$ coincides with a natural algebraic filtration based on the kernels of restrictions to subgroups. This is a very natural algebraic condition, leading to the following conjecture.

Conjecture 3.1. *The natural filtration by kernels of restriction maps on $\underline{M}$ coincides with the effect in homotopy of the slice filtration on $H\underline{M}$ for all finite G.*

In fact, the proof given in the paper applies to any group for which all subgroups are normal.

The proof of the equivalence of the two natural filtrations on $H\underline{M}$ relied on an understanding of the interplay of the slices of G/N-spectra and the slices of G-spectra. There is a natural "pull-back" map which embeds G/N-spectra into G-spectra [38]. Natural adjunctions similar to those in Greenlees-May give bounds on the relative slice dimensions [21].

Theorem 4. *If k is 0 or 1 modulo $|G/N|$, then $(k-1)$-slices for G/N pull back to $(k|N|-1)$-slices for G.*

For Eilenberg-Mac Lane spectra $H\underline{M}$, this was sufficient to show that the slice filtration on $H\underline{M}$ and the algebraic filtration on $\underline{M}$ agree, since all of the resulting filtration quotients are pulled-back 0-slices. However, for other dimensions, I could only provide bounds on the slice dimensions of pulled-back slices.

Conjecture 3.2. *The $(k-1)$-slices for G/N pull back to $(k|N|-1)$-slices for G for all k.*

On particularly nice application for the previous theorem is when $N = G$. In this case, every integer is zero or one modulo $|G/N|$. The slice tower for G/G-spectra is just the ordinary Postnikov tower. Since the slice tower then pulls back to the slice tower, we see that for spectra whose H-fixed points are contractible for proper subgroups H, the slice tower is simply a reindexed form of the Postnikov tower. All equivariant spectra have a filtration which carves out these geometric fixed point pieces (this is the isotropy separation sequence), and the slice tower for a general spectrum then reassembles these sheared Postnikov towers for the various geometric fixed points.

3.2 The homotopy groups of Hopkins-Miller higher real K-theories

Fix a perfect field k of characteristic $p > 0$, and let F be a formal group over k. Then there is a complex orientable spectrum $E(k, F)$ whose associated formal group is the universal deformation of F, the coefficients of which are

$$\pi_* E(k, F) \cong \mathbb{W}(k)[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}],$$

where n is the height of the formal group and $\mathbb{W}$ is the Witt vector functor.

The Hopkins-Miller theorem says that $E(-,-)$ is actually a functor from perfect fields and formal groups of finite height to E_∞-ring spectra [48, 20], and if $k = \mathbb{F}_{p^n}$ and F is the Honda formal group law of height n, then we denote $E(k, F)$ by E_n and call it the Lubin-Tate spectrum. By the Hopkins-Miller theorem, this is an E_∞-ring spectrum on which the Morava stabilizer group acts by E_∞-ring maps. In particular, for finite subgroups G, we can define the "higher real K-theory spectra" $EO_n(G)$ as

$$EO_n(G) = E_n^{hG}.$$

These are again E_∞-ring spectra. Work of Adams-Baird, Ravenel, Behrens, Goerss-Henn-Mahowald-Rezk, and Behrens-Lawson has shown that the spectra $EO_n(G)$ for various G serve as nice approximations to the $K(n)$-local sphere, assembling into diagrams that approximate or directly produce this localization [47], [5], [19], [6].

For $p = 2$, many of these spectra have more familiar names. For $n = 1$ and $G = C_2$, $EO_1(C_2)$ is the 2-completion of KO, and for $n = 2$ and $G = Q_8 \rtimes C_3$, the semi-direct product of the quaternions with C_3, $EO_2(G)$ is the 2-completion of TMF, the periodic spectrum of topological modular forms. Finally, for any n and $G = C_2$, Averett has shown that $EO_n(C_2)$ is the $K(n)$-localization of the fixed points of the Real Johnson-Wilson spectrum $E\mathbb{R}(n)$ [4].

The spectra $EO_n(G)$ have the added advantage of being relatively computable. The Adams-Novikov spectral sequence for $EO_n(G)$ is the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s\big(G; \pi_t(E_n)\big) \Rightarrow \pi_{t-s} EO_n(G).$$

The Nilpotence Theorem shows that although this spectral sequence is an upper-half plane spectral sequence, it converges strongly with a horizontal vanishing line [14]. In particular, there are only finitely many differentials to determine. These are very difficult to determine in practice.

When $n = (p - 1)$ and $G = C_p$, Hopkins and Miller computed the differentials in this spectral sequence, determining the torsion in the homotopy groups of $EO_{p-1}(C_p)$ [31]. This computation provides an initial connection to the Kervaire problem and was instrumental to our thinking, since it provides a rigidification of (and correction to) Ravenel's original approach to the odd-primary Kervaire invariant one problem [46]. In an earlier collaborative effort, Hopkins, Ravenel, and I worked out the homotopy fixed point differentials for $EO_{f(p-1)}(C_p)$, but we had little success for larger groups G.

Goal 3.3. *Use the new equivariant approaches and machinery to revisit the homotopy of $EO_n(G)$, deriving differentials in the homotopy fixed point spectral sequence and determining extensions.*

Since the slice filtration is one of equivariant spectra, we must first take E_n as a naive spectrum with G-action and produce a genuine one. This is easily accomplished by simply pushing it forward to genuine spectra (or equivalently, smashing with the genuine S^0). The Nilpotence Theorem shows that for all non-trivial subgroups H of G, the Tate spectrum E_n^{tH} is contractible. This means we can actually produce a G-commutative ring spectrum for which the fixed and homotopy fixed points agree and are $EO_n(G)$.

While conceptually similar, the existence of $MU_{\mathbb{R}}$ shows that the approaches are very different for $p = 2$ and for $p > 2$. All of this work will be joint with Hopkins and Ravenel.

3.2.1 $EO_n(G)$ at 2 and the equivariant homotopy groups of $MU^{(G)}$

One way to express Averett's result is that there is a Real orientation of E_n viewed as a C_2-spectrum.

Conjecture 3.4. *Averett's orientation can be refined to a C_2-equivariant commutative ring map $MU_{\mathbb{R}} \to E_n$.*

Since the norm is the left adjoint to the forgetful functor on equivariant commutative rings, if C_{2^m} acts on E_n (which simply requires that 2^{m-1} divides n), then there is an equivariant commutative ring map

$$N_{C_2}^{C_{2^m}} MU_{\mathbb{R}} \to E_n.$$

This means we can make tremendous progress by computing the equivariant homotopy groups of $MU^{(G)}$, and this is accessible through the slice filtration. For $G = C_2$, the slice differentials also recover all of the homotopy fixed point differentials, and for larger groups, we will have similar results by the connection between the slice and homotopy fixed point spectral sequences.

In Hill-Hopkins-Ravenel, we determined a large family of differentials based only on elementary considerations of the geometric fixed points. This completely determined the orientability of sign representation spheres and their norms for $MU^{(G)}$-modules, and the differentials were sufficient to determine the required 256-fold periodicity by explicitly producing permanent cycles. However, even for C_4, we did not completely determine all differentials and extensions in the spectral sequence.

Goal 3.5. *Determine the remaining differentials and the Mackey functor homotopy groups of $MU^{(G)}$.*

In particular, there is a natural subgoal of particular interest.

Goal 3.6. *Compute $\pi_{126}\Omega_{\mathbb{O}}^{hC_8}$ and see if it settles the remaining Kervaire case.*

I have several observations and conjectures about what happens in the slice filtration, and these make me optimistic that we can completely understand the slice spectral sequence for $MU^{(G)}$.

On the computational front, the fact that this is a spectral sequence of Mackey functors imposes tremendous constraints. The underlying homotopy is just that of an $(|G|/2)$-fold smash product of MU, and the homotopy groups of the C_2-fixed points were essentially determined by Araki and Hu-Kriz [3], [32]. In particular, the Hurewicz image in $\pi_* MU_{\mathbb{R}}^{C_2}$ contains η, ν, and σ, but with the exception of η, none occur in the "expected" slice filtration of 1. The fact that ν occurs in filtration 3 for the C_2 fixed points shows, for example, that in the C_4 fixed points, ν is detected in filtration 1 and that 2ν is a non-zero filtration 3 element. Similar, though trickier, computations show that for $G = C_8$, σ is detected on the 1-line and 4σ is a non-zero element (of filtration 7). Thus the filtration on which an element is detected can drop as the order of the group increases.

While increasing the order of the group does increase the possible orders which appear on torsion elements (transfer arguments showing that the order of the group is an absolute upper bound on the torsion exponent of any torsion element), the connection seems to weaken as we approach the order of key elements. For example, it seems that all of the low dimensional differentials, for any cyclic 2-group G, conspire to ensure that $\eta^3 = 4\nu$ is always zero in the G-fixed points. Similarly, conjectural computations show that 8σ is also always zero (which in turn means that we have an element v_1^4 in the G-fixed points for all G).

Conjecture 3.7. *For elements in the Hurewicz image, the order they have when they first appear in their expected (the Adams-Novikov) filtration is the largest order they ever achieve.*

As $|G|$ increases, so should the Hurewicz image, and we therefore have a pro-system of commutative rings under $\pi_* S^0$ (in fact, a pro-system of homotopy groups of commutative ring spectra, meaning we have compatibility with power operations). The inverse limit is therefore an interesting (and at least in low degrees very approachable) quotient of the stable homotopy groups of spheres. Since, the Hurewicz image for the G-fixed points serves as an upper bound for the Hurewicz image for $EO_n(G)$, this limit (and its finite stages) provides essential input for the $EO_n(G)$ computations.

There is another, more direct connection, at this point supported only by toy models. The underlying homotopy of $MU^{(G)}$ is polynomial on $(|G|/2)$-classes in every even dimension. For easy of notation, we will denote the $(|G|/2)$ generators in degree $2i$ by $G \cdot r_i$. There are also identical versions built out of BP, though these are not known to be E_∞. The classes r_i refine to C_2-equivariant maps $\bar{r}_i\colon S^{i\rho_2} \to MU^{(G)}$, and similarly for $BP^{(G)}$. As described in Hill-Hopkins-Ravenel, we can form $MU^{(G)}$-module spectra

$$BP\langle f\rangle^{(G)} = BP^{(G)}/(G \cdot \bar{r}_{2^j-1}, j > f).$$

If $G = C_2$, then $BP\langle f\rangle^{(G)}$ is the Real Johnson-Wilson theory $BP\langle f\rangle_{\mathbb{R}}$ constructed by Hu-Kriz and studied Kitchloo-Wilson [32], [36], [37]. For G larger than C_2, then the spectrum $BP\langle f\rangle^{(G)}$ is not underlain by $BP\langle f\rangle$ but rather by an algebraic extension of $BP\langle f \cdot |G|/2\rangle$, and hence it sees increasingly more chromatic phenomena as $|G|$ increases.

Empirical evidence suggests that the conjectured equivariant map $MU^{(G)} \to E_n$ actually factors through $BP\langle f\rangle^{(G)}$ if $n = f \cdot |G|/2$. Moreover, the passage from $BP\langle f\rangle^{(G)}$ to E_n seems to be a composite of two functors: invert the norm of r_{2^f-1}, thereby passing to $E(f)^{(G)}$, and then $K(n)$-localize. Thus $BP\langle f\rangle^{(G)}$ serves as a connective, global G-equivariant approximation to E_n. As an added advantage, the slices of $BP\langle f\rangle^{(G)}$ are readily determined by those of $MU^{(G)}$, and the slice differentials follow immediately from those on $MU^{(G)}$. The chief downside to this approach is that the passage from $MU^{(G)}$ to $BP\langle f\rangle^{(G)}$ is not the killing of a regular sequence, and we therefore do not even know that $BP\langle f\rangle^{(G)}$ is A_∞.

Goal 3.8. *Determine when an $RO(G)$-graded sequence of homotopy elements in a commutative ring spectrum is sufficiently "regular" to conclude that the quotient is associative.*

Our toy computations again show no obvious obstructions, and for $G = C_2$ and C_4 and for $f = 1$, the resulting spectrum is actually E_∞.

3.3 The Odd Primary Slice Filtration

At odd primes, the situation is much harder. The chief difficulty is a lack of odd-primary analogue for $MU_{\mathbb{R}}$.

Goal 3.9. *Build an odd-primary analogue of $MU_{\mathbb{R}}$, unlocking many of the tools described at 2 and for the Kervaire problem.*

An immediate application would be the resolution of the 3-primary Kervaire invariant one problem: the survival of the $\beta_{3^i/3^i}$ family.

There is a great deal of anecdotal evidence supporting understanding the slice spectral sequence for $EO_n(G)$ at an odd prime.

In earlier, unpublished work, Hopkins, Ravenel and I showed that the homotopy of E_n admitted a particularly simple description as a G-algebra. The key fact was that for $n = p^k(p-1)$ and $G = C_{p^{k+1}}$, the homotopy looks like a completion of a localization of a polynomial algebra on $p^k(p-1)$ generators, all of which are essentially permuted by the group (for larger n divisible by this choice, we simply tensor together several copies), a description exactly analogous to the one just described for $p = 2$. The first piece of evidence supporting such a simple story is that there is an equivariant spectrum W such that the underlying homotopy type is a wedge of $p^k(p-1)$ spheres of dimension exactly the dimension of the polynomial generators, and upon smashing W with $H\underline{\mathbb{Z}}$, we get a slice of the appropriate dimension (just as at the prime 2).

Building on this, Hopkins, Ravenel, and I have a toy model for something like the slice spectral sequence, together with all of the differentials. This guess is formally similar to the case of $p = 2$, in that it comes from a global version and the geometric fixed points have a prescribed form. The second piece of evidence is that based only on these two assumptions, we recover all differentials we worked out for $G = C_p$ and $n = f(p-1)$. This analysis only uses equivariant machinery and language, so in particular, the methods used give an upper bound on the vanishing lines in the odd-primary homotopy fixed point spectral sequences for $EO_n(G)$. We believe this gives a coarse upper bound for the analogous 3-primary Kervaire invariant one problem.

All progress is hampered by a lack of a "complex orienting" spectrum. With an odd primary $MU_{\mathbb{R}}$, much of what happens at 2 could simply be copied. Ideally, we could use geometry to produce a nice, commutative model, but sadly, the obvious guesses all have the wrong equivariant homotopy type. One approach to this problem is to build the relevant analogue of $MU_{\mathbb{R}}$ by hand. Here there is tremendous support from formal group data. If we forgot about manifolds, then $MU_{\mathbb{R}}$ is classifying a formal group law together with an automorphism (the $[-1]$-series). The passage to geometric fixed points amounts the quotient by the group action, and here that results in imposing a relation $[2](x) = 0$ on the formal group. This yields Quillen's description of the formal group carried by MO [45, 44]. These should be redoable for an odd prime. For $p > 2$, we want a collection of $(p-1)$-formal groups and isomorphisms between them which are in some sense traceless (one way to impose this is to look at higher dimensional formal groups which non-canonically split into 1-dimensional ones). The underlying spectrum is then $MU^{\wedge(p-1)}$ with a curious C_p-action. The geometric fixed points are now the spectrum carrying the universal formal group in which $[p]$ is zero, and just as for $p = 2$, this is a wedge of Eilenberg-Mac Lane spectra (indexed by the non-v_i-generators of MU).

This description gives us another, possibly more direct, construction. We need only find a naive C_p-action on $MU^{\wedge(p-1)}$ or more easily on $BP^{\wedge(p-1)}$. Since C_p has only two subgroups, it is easy to describe a C_p equivariant spectrum: the isotropy separation sequence shows us that we need only specify a naive C_p spectrum, the desired geometric fixed points, and a "completion map" from the desired geometric fixed points to the Tate spectrum of the naive C_p-spectrum. For $BP^{\wedge(p-1)}$, we want the geometric fixed points to be $H\mathbb{F}_p$, and so we need a C_p action for which the Tate spectrum is an $H\mathbb{F}_p$-module. It is not difficult to write out a Tate spectral sequence on $BP^{\wedge(p-1)}$ together with a pattern of differentials which produces $H\mathbb{F}_p$ (and in fact, the differentials then again imply all of the homotopy fixed point differentials for $EO_{f(p-1)}(C_p)$ for all f), but this requires a C_p-action which is at the very least correct on homotopy groups.

Goal 3.10. *Produce a C_p-action on $BP^{\wedge(p-1)}$ for which the Tate spectrum is $H\mathbb{F}_p$.*

4 New Approaches in Topological Hochschild Homology

4.1 Topological Hochschild Homology and the norm

Angeltveit, Blumberg, Gerhardt, Lawson, and I are using the Hill-Hopkins-Ravenel norm technology to produce a new, conceptually simpler model for topological Hochschild homology. The model arose from a simple observation: when viewed H-equivariantly, the standard cyclic bar model for $THH(R)$ is built out of $N_e^H(R)$. This lead to a fairly straightforward definition for $THH(R)$ as an S^1-equivariant spectrum: the left adjoint to the forgetful functor from S^1-equivariant commutative rings to ordinary commutative rings. For sake of comparison, let $T(-)$ denote this left adjoint.

Goal 4.1. *Show that $T(R)$ is equivalent to* $\mathrm{THH}(R)$ *as a cyclotomic spectrum and explore the computational and theoretical applications.*

The unit map $R \to \mathrm{THH}(R)$ is a commutative ring map, and therefore corresponds to an S^1-equivariant map $T(R) \to \mathrm{THH}(R)$. It is not difficult to show that this map is an H-equivariant equivalence for all finite H. The key step here is that "multiplication by h" is an H commutative ring map $N^H R \to N^H R$ and this gives rise to a "tensor product over H". The Bökstedt-Madsen model then has

$$\mathrm{THH}(R) = S^1 \otimes_H N_e^H(R),$$

and it is not difficult to compare this to the left adjoint. This model is also visibly cyclotomic, and the identifications with the Bökstedt-Madsen model respect this structure [9]. The Bökstedt-Hsiang-Madsen TC and the Hesselholt-Madsen TR-tower care not about the actual S^1-equivariant homotopy type but rather about a "compatible C_n-homotopy type for all n". From this perspective, our model of THH is interchangable with the Bökstedt-Madsen one.

In fact, we do not know if the S^1-fixed points (or equivalently, the S^1-geometric fixed points) for this version of THH agree with those of the Bökstedt-Madsen model. We suspect that general properties of this left adjoint construction show that the S^1-geometric fixed points of $THH(R)$ are just R, which differs from the standard constructions. This inequivalence of S^1-fixed points implies that $T(R)$ is not S^1-equivariantly equivalent to Bökstedt's $\mathrm{THH}(R)$.

Being a left adjoint, $T(R)$ has very nice formal properties. A useful application of this new model is a genuine S^1-equivariant model for THH of a Thom spectrum M_f generalizing the Blumberg-Cohen-Schlichtkrull non-equivariant model and generalizing the Madsen model for suspension spectra [7], [41].

Goal 4.2. *Describe a new model for* THH *of a Thom spectrum as a cyclotomic spectrum.*

This suggests that it will have similar computational benefits for approaching the algebraic K-theory of Thom spectra. Additional applications are two natural relative versions described below of both THH and the TR tower.

4.2 Computations with equivariant THH and Thom Spectra

One of the most exciting aspects of the joint project with Angeltveit, Blumberg, Gerhardt, and Lawson on a new equivariant model for THH is the possibility for various flavors of relative versions. In particular, we can perform two kinds of constructions:

1. THH_G for G-equivariant ring spectra, where G is a subgroup of S^1, and
2. THH^R in the category of R-modules for a fixed commutative ring spectrum R.

The aim of this project (joint with the above named collaborators) is to explore both aspects.

4.2.1 $THH_{C_2}(MU_{\mathbb{R}})$

Since MU is a Thom spectrum, our joint work provides an S^1-equivariant description of $\mathrm{THH}(MU)$ analogous to the Blumberg-Cohen-Schlichtkrull one [7]. The spectrum $\mathrm{THH}(MU)$ provides a fantastic and important initial example for all complex orientable theories (and probably recording essentially complete information for Landweber exact homology theories). However, even with our description the homotopy groups of the fixed points will be difficult to determine. For example, if $H \subset S^1$ is a finite subgroup, then there is a natural H-equivariant map

$$N_e^H MU \to \mathrm{THH}(MU),$$

which generalizes the non-equivariant unit map $MU \to \mathrm{THH}(MU)$. Even in π_0 do complications arise: $\underline{\pi_0}(N_e^H MU)$ is the Burnside Mackey functor for H. This suggests that the fixed points will become increasingly difficult to compute.

If we replace MU with $MU_{\mathbb{R}}$ and replace THH with a relative version, then we have a less universal example which is still very interesting. The relative model for THH here is THH_{C_2}, the "left adjoint to the forgetful functor from commutative S^1-ring spectra to commutative C_2-ring spectra". We want to think of $\mathrm{THH}_{C_2}(MU_{\mathbb{R}})$ as $S^1 \otimes_{C_2} MU_{\mathbb{R}}$ to $\mathrm{THH}(MU)$'s $S^1 \otimes MU$. The canonical C_2 equivariant map $N_e^{C_2} MU \to MU_{\mathbb{R}}$ induces a natural S^1-equivariant map

$$\mathrm{THH}(MU) = S^1 \otimes_{C_2} N_{\{e\}}^{C_2} MU_{\mathbb{R}} \to S^1 \otimes_{C_2} MU_{\mathbb{R}},$$

and the natural map from $\mathrm{THH}(MU)$ to $\mathrm{THH}(R)$ for R a Real oriented theory will descend to this relative form. Since many naturally occurring spectra are Real oriented, this provides a potentially simpler intermediary.

Goal 4.3. *Determine the equivariant homotopy groups of* $\mathrm{THH}_{C_2}(MU_{\mathbb{R}})$.

This relative version will be much more computable than the absolute version. In particular, the unit map from $N_e^H MU$ is replaced with a map

$$N_{C_2}^H MU_{\mathbb{R}} \to S^1 \otimes_{C_2} MU_{\mathbb{R}}.$$

Even on π_0 the story is much simpler: $\underline{\pi_0} MU^{(H)}$ is the constant Mackey functor $\underline{\mathbb{Z}}$ rather than the Burnside ring. Thus for this relative THH all transfer maps automatically act as multiplication by the index of one subgroup in another.

This equivariant unit map also allows us to apply our slice spectral sequence computations for the spectra $MU^{(H)}$, immediately giving information about the homotopy groups of $\mathrm{THH}_{C_2}(MU_{\mathbb{R}})$. In particular, we know that the Hurewicz image for the H-fixed points factors through that of $MU^{(H)}$. This means that the Hurewicz image of a spectrum closely related to the relative $TR(MU)$, the spectrum $TF(MU_{\mathbb{R}})$ (defined just as in the absolute case), will factor through the inverse limit described above.

Moreover, I suspect that the computation will be accessible via slice-theoretic methods.

Conjecture 4.4. *The slices of* $\mathrm{THH}_{C_2}(MU_{\mathbb{R}})$ *are as simple as those of the norms of* $MU_{\mathbb{R}}$*.*

While they will no longer be concentrated in even dimensions, they should all be induced regular representation spheres from finite subgroups of S^1. In both the case of $MU^{(H)}$ and this relative THH, the key issues for computing the slices are a control of the underlying homotopy and of $\underline{\pi_0}$. A slice filtration of this form would make computations very straightforward.

4.2.2 Relative THH

The second flavor of relative THH is THH in a category of R-modules for a fixed R algebra. This project is very promising but much more speculative than the previous ones. As described above, the norm functor really requires only a symmetric monoidal category, arising from the interplay of indexing categories for diagram categories.

Goal 4.5. *Form an* S^1*-equivariant model of* THH *in the category of* R*-algebras.*

We already know the underlying homotopy type that we want. The cyclic bar construction used by Bökstedt to define THH applies equally well to define THH^R on the category of R-algebras. The problem is that this does not work well with the S^1-equivariance. In particular, we have no way to apply this sort of approach to a relative TC a la Bökstedt-Hsiang-Madsen.

The norm machinery seems to provide a way around this. The most obvious approach is to simply use the norm in H-spectra (forgetting for the time being the R-module structure). Being a symmetric monoidal functor, this takes R-modules to $N_H^G(R)$-modules. Ignoring for the moment issues of cofibrancy, this construction has the desired interaction with geometric fixed points (potentially resulting in a cyclotomic spectrum). This, however, has replaced R-modules with a different category since we have neglected structure. To get the functor we want, we must return to R-modules while remembering the equivariance.

The norm will of course always change categories: it takes an H-indexed diagram category to a G-indexed diagram category. This approach is more in line with what we want: H-indexed diagrams in R-modules should yield G-indexed diagrams in R-modules. For universal reasons, the basic example to understand is R as an R-algebra. The simplest example is $G = C_2$. Here the symmetric monoidal product is smash over R, and the norm from the trivial group to C_2 is underlain by $R \wedge_R R$. The C_2 action flips the two factors, and we see that as a commutative C_2 ring spectrum, we are looking at $R \wedge S^0$.

For a general G, the same will be true for the norm of R in the category of R-modules, and thus the norm is a functor from R-modules to G-equivariant $(R \wedge S^0)$-modules. Again ignoring cofibrancy issues, these two norms explain how to determine the relative one. Since $R \wedge S^0$ is a G-commutative ring spectrum, it receives a map $N_e^G R \to R \wedge S^0$. Base-changing $N_e^G M$ along this map realizes the norm in R-modules.

Verifying the homotopical properties will allow us to mirror our construction of THH in this relative context. Thus our THH will be the left-adjoint to the forgetful functor from S^1-equivariant $R \wedge S^0$-algebras to R-algebras, and for finite subgroups H of S^1, this will be identified with $S^1 \otimes_H N_e^H(-)$ (where our tensoring operation takes place in R-algebras). The formal properties of the interaction of the norm and geometric fixed points will ensure that the resulting spectrum is cyclotomic, and we can therefore copy the TR and TC machinery to produce R-module spectra $TR^R(A)$ and $TC^R(A)$. These will be approximations to the yet unconstructed relative K-theory spectrum, and this will help provide intuition for how to understand this relative K-theory.

An illuminating example will be to work in the category of $H\underline{\mathbb{Z}}$-algebras. This category is algebraic, and an understanding of the relative norm in this category should produce interesting computations. We hope that my student Kristen Mazur's on-going work determining the homotopy groups of $N_H^G(R)$ for an H-spectrum R will make this an approachable computation.

4.3 Equivariant THH and Red-shift

Angeltveit and Gerhardt, generalizing work of Hesselholt-Madsen, computed the $RO(S^1)$-graded equivariant homotopy groups of THH($\mathbb{F}_p$) [1], [25]. Since $RO(S^1)$ is the natural indexing set for equivariant homotopy, the eventual goal is to find an algebraic description of these groups, similar to the Hesselholt-Madsen deRham-Witt vector formulation for the integer graded homotopy. My recent work with equivariant computations has given me facility with computing equivariant homology and cohomology with coefficients in a general Mackey functor, and Gerhardt and I decided to apply this to reassess the Angeltveit-Gerhardt result via the equivariant Atiyah-Hirzebruch spectral sequence.

The computation is actually surprisingly simple: the Mackey functor homotopy groups of THH($\mathbb{F}_p$) and the linear ordering of subgroups of C_{p^k} conspire to produce a very sparse result for any representation sphere. Taking as input the Hesselholt computations of TC of truncated polynomial algebras [23], we produced a family of d_3 differentials and companion extensions. These are exactly the same as the extensions and differentials observed in the non-equivariant Atiyah-Hirzebruch spectral sequence for the topological K-theory of spectra like $\mathbb{R}P^n$! This observation recovers Rognes's results about chromatic red shift in algebraic K-theory (since it means the k-invariant is $Q_1 \bmod p^n$), but it is unclear exactly how THH is producing the v_1 elements. In general, it is difficult to find naturally occurring commutative ring spectra of finite chromatic heights greater than 2. Rognes's red shift immediately produces whole families simply by iterating algebraic K-theory or by analyzing THH.

Our approach also provides a foil to the more traditional approaches to computing the $RO(S^1)$-graded homotopy, in that it provides a direct connection between the geometric, isotropy separation methods employed by Angeltveit-Gerhardt and homotopical data from the Postnikov tower as recorded by differentials in the Atiyah-Hirzebruch spectral sequence. Gerhardt and I intend to continue exploring this interplay.

Goal 4.6. *Sketch analogues of these results for $H\underline{\mathbb{Z}}$ and ℓ, and then attempt ko and tmf.*

The $RO(G)$-graded equivariant homotopy groups of THH($H\underline{\mathbb{Z}}$) and THH(ℓ) were also determined, with finite coefficients, by Angeltveit-Gerhardt, and for *ko* and *tmf*, the starting points are computations of Angeltveit-Hill-Lawson and Bruner-Rognes [2], [13].

The chief difficulty of this approach is that we do not have even a rudimentary understanding of the Steenrod algebra for Mackey functors which are not fields in the sense of Lewis [39]. For

Mackey fields, Oruç has shown that the Steenrod algebra in this case is the ordinary one twisted by the endomorphisms of the Mackey functor [43]. Sadly, the Mackey functors which arise in THH are not Mackey fields, and the resolution of them by fields is difficult. For G a cyclic 2 group, the spectrum $BP\langle 0\rangle^{(G)}$ described above is $H\underline{\mathbb{Z}}$, and this provides a method for computing $H\underline{\mathbb{Z}}\wedge H\underline{\mathbb{Z}}$ and from this $H\underline{\mathbb{F}_2}\wedge H\underline{\mathbb{F}_2}$, recovering the result of Hu-Kriz for $G=C_2$ [32], but for larger groups, this gives a difficult to use description at best.

Goal 4.7. *Identify and construct in an elementary way natural elements corresponding to the Milnor primitives Q_i.*

5 Broader Impact

The broader impact of my project centers around three major foci: working with early-career mathematicians and underrepresented groups, providing opportunities for mathematical discussion through major conference organization, and exploring new technology to help reach people outside of my field, including non-mathematical audiences.

My collaborators come from all different stages of their careers, including many who are starting their careers. Tyler Lawson received his PhD in 2004, Vigleik Angeltveit received his PhD in 2006, and Teena Gerhardt received her PhD in 2007. Additionally, I have three graduate students: Kristen Mazur, Calder Wishne, and Carolyn Yarnall. I have very much enjoyed the experience of advising and intend to continue working directly with graduate students over the course of the project. My work with my two female graduate students, Kristen and Carolyn, also underscores my commitment to underrepresented groups in math. I directly mentor them as their advisor, but cognizant of my limitations helping them navigate the waters of being women in mathematics, I have striven to introduce them to more senior women in the field.

While at UVA, I have run working groups and seminars to help introduce graduate students and undergraduates to some of the approaches to computational homotopy theory. I taught a graduate course on spectral sequences in algebraic topology, posting notes and recorded lectures online for anyone to access, and I have corresponded with a number of people who have used these with their own students or for their own research. I intend to supplement this with additional short notes covering basic topics in the field, building a small library of freely accessible material running the gamut of computational techniques in algebraic topology.

Since taking over as the primary organizer for the topology seminar at UVA, I have transformed it into an active forum for outside speakers. We have especially stressed inviting early-career mathematicians, especially ones who are on the job market. For the last three semesters, we have averaged about 5 young speakers (about a third of whom are women), and we have a large number signed-up for next semester. This influx of outside speakers benefits everyone: the speaker, the local graduate students, and the topology faculty have all greatly enjoyed speaking about new ideas.

During the next three years, I am co-organizing three international conferences: one at Banff in February 2012 (“Algebraic K-theory and Equivariant Homotopy Theory”), one at UVA in June 2012, and the first semester long program in algebraic topology in 25 years at MSRI in Spring 2014. These programs will be fantastic opportunities for broader dissemination of mathematical concepts (both directly related to my proposal and in the entire field), and they will undoubtedly result in new working relationships and collaborations. We have also worked diligently with both Banff and MSRI to ensure the participation of underrepresented groups, and with MSRI, we have crafted explicit plans for a workshop for women and for recruiting from institutions historically associated with people of color.

MSRI also presents an opportunity to merge my conference organizing with my work with early career mathematicians. Teena Gerhardt and I intend to run a "graduate student retreat" at MSRI. This will help students (and early career post-docs) navigate the confusing waters of applying for jobs, transitioning from graduate school, and writing grants. Gerhardt and I will co-facilitate, bringing together more senior people in the field to work with panels and small groups, answering questions and providing advice.

Finally, I am also interested in ways to reach a broader audience. After we announced the solution to the Kervaire problem, Hopkins, Ravenel, and I met with Dana Mackenzie, a freelance mathematical writer who described our work for a mathematically savvy audience in "What's Happening in the Mathematical Sciences" [40]. This discussion stressed for me the importance and difficulty of explaining results to lay audiences. Finding appropriate toy models which allowed people to understand metaphorically or metonymically the work we did was challenging but also forced us to fit our work into a broader tapestry of mathematics. I intend to continue this, writing and recording as video podcasts advanced mathematical concepts for an audience of non-specialists. Technology like iTunesU and YouTube make distribution of recordings, PDFs, and videos easy, and I think both interested non-mathematicians and working researchers could benefit from this material.

6 Results from Prior NSF Support

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During this grant, all of the papers related to the Hill-Hopkins-Ravenel solution to the Kervaire invariant one problem were written. While our solution was announced prior to the receipt of the grant, the write-up required substantial new mathematics, all of which was done during the period covered by the grant [29]. Additionally, two survey articles, one covering the history and one covering the proof, were written for Harvard's "Current Developments in Mathematics" lecture series [30], [28]. The article about the proof also included several independent arguments for key parts of the solution, providing additional intuition and other approaches. The Slice Primer described in Section 3.1 was also worked out, written, and submitted during this time [26].

I also wrote and published my paper computing Ext over the motivic analogue of the subalgebra $\mathcal{A}(1)$ of the Steenrod algebra over the field $\mathbb{R}$ [27]. This introduced my "ρ-Bockstein spectral sequence" used extensively now by Isaksen in his computations with the motivic Adams spectral sequence. The computation also provides a close connection to the motivic image of J (and $K(1)$-local phenomena), and it recovers the computation of Suslin on the algebraic K-theory of $\mathbb{R}$ [49].

While supported by these grants, I gave 14 talks domestically. I also gave 10 talks at international conferences, including a series of 3 lectures at the Oberwolfach topology meeting in September 2010.