Worksheet 06 – Solutions

Instructor: Hong Chen **Topics:** Sections 3.3, 3.5, 3.7–3.9

1. Implicit differentiation

(a) $x^2 + xy + y^2 = 7$ **at** $(1, 2)$

Treat y as a function of x and differentiate both sides:

$$2x + y + xy' + 2yy' = 0.$$

Collect the terms containing y':

$$(x + 2y)y' = -(2x + y), \qquad \boxed{y' = -\frac{2x + y}{x + 2y}}, \quad x + 2y \neq 0.$$

At $(1, 2)$, the denominator is $5 \neq 0$, giving $\boxed{y' = -4/5}$.

(b) $xe^y + y^2 = 1$ **at** $(1, 0)$

The product rule and chain rule give

$$e^y + xe^y y' + 2yy' = 0.$$

Therefore

$$(xe^y + 2y)y' = -e^y, \qquad \boxed{y' = -\frac{e^y}{xe^y + 2y}}, \quad xe^y + 2y \neq 0.$$

At $(1, 0)$, the denominator is $1 \neq 0$, giving $\boxed{y' = -1}$.

2. Inverse trigonometric functions

(a) $f(x) = x^2 \cdot \arctan(3x)$

Use the product rule and then the chain rule for arctangent:

$$f'(x) = 2x \cdot \arctan(3x) + x^2 \cdot \frac{1}{1 + (3x)^2} \cdot 3$$

$$= \boxed{2x \arctan(3x) + \frac{3x^2}{1 + 9x^2}}.$$

This formula holds for every $x \in \mathbb{R}$.

(b) $g(x) = \arcsin(x^2/2)$

By the chain rule,

$$g'(x) = \frac{1}{\sqrt{1 - (x^2/2)^2}} \cdot x = \boxed{\frac{2x}{\sqrt{4 - x^4}}}.$$

The function requires $-1 \leq x^2/2 \leq 1$, so its domain is $[-\sqrt{2}, \sqrt{2}]$. The derivative formula holds on $\boxed{(-\sqrt{2}, \sqrt{2})}$, where the denominator is positive.

3. Products and compositions

(a) $y = (x^3 - 1) \cdot e^{\sin x} \cdot \arctan x$

Differentiate one factor in each term, retaining the inner derivative of $e^{\sin x}$:

$$\begin{aligned} y' &= 3x^2 \cdot e^{\sin x} \cdot \arctan x \\ &\quad + (x^3 - 1) \cdot e^{\sin x} \cdot \cos x \cdot \arctan x \\ &\quad + (x^3 - 1) \cdot e^{\sin x} \cdot \frac{1}{1 + x^2}. \end{aligned}$$

This formula holds for every $x \in \mathbb{R}$.

(b) $y = \sin(e^{x^2+1})$

The layers are sine, the exponential, and $x^2 + 1$. The chain rule gives

$$y' = \cos(e^{x^2+1}) \cdot e^{x^2+1} \cdot 2x = \boxed{2x \cdot e^{x^2+1} \cdot \cos(e^{x^2+1})}.$$

This formula holds for every $x \in \mathbb{R}$.

4. Exponential and logarithmic functions

(a) $f(x) = 2^{\cos x}$

The exponential rule for base 2 and the chain rule give

$$f'(x) = 2^{\cos x} \cdot \ln 2 \cdot (-\sin x) = \boxed{-(\ln 2) \cdot 2^{\cos x} \cdot \sin x}.$$

(b) $g(x) = \log_3(1 + e^{2x})$

The logarithmic rule for base 3 and the chain rule give

$$g'(x) = \frac{1}{(1 + e^{2x}) \ln 3} \cdot 2e^{2x} = \boxed{\frac{2e^{2x}}{(1 + e^{2x}) \ln 3}}.$$

Both formulas hold for every real x, since $1 + e^{2x} > 0$.

5. First and second derivatives

For $f(x) = \ln(1 + e^x)$, the chain rule gives

$$\boxed{f'(x) = \frac{e^x}{1 + e^x}}.$$

Differentiate this quotient:

$$f''(x) = \frac{e^x(1 + e^x) - e^x \cdot e^x}{(1 + e^x)^2} = \boxed{\frac{e^x}{(1 + e^x)^2}}.$$

These formulas hold for every real x. At $x = 0$, $\boxed{f''(0) = 1/(1 + 1)^2 = 1/4}$.