Worksheet 05 – Solutions

Instructor: Hong Chen **Topics:** Sections 3.5–3.8

1. Chain rule and trigonometric derivatives

(a) $f(x) = \dfrac{1}{\sqrt{1+x^4}}$

Write $f(x) = (1+x^4)^{-1/2}$ and apply the chain rule:

$$f'(x) = -\frac{1}{2}(1+x^4)^{-3/2}(4x^3) = \boxed{-\frac{2x^3}{(1+x^4)^{3/2}}}.$$

(b) $g(x) = e^{\cos(2x)}$

$$g'(x) = e^{\cos(2x)}(-\sin(2x))(2) = \boxed{-2e^{\cos(2x)}\sin(2x)}.$$

(c) $h(x) = x\sin(3x)$

$$h'(x) = 1\cdot\sin(3x) + x\cos(3x)\cdot 3 = \boxed{\sin(3x) + 3x\cos(3x)}.$$

(d) $k(x) = \dfrac{\cos x}{1+x^2}$

$$k'(x) = \frac{(1+x^2)(-\sin x) - (\cos x)(2x)}{(1+x^2)^2} = \boxed{\frac{-(1+x^2)\sin x - 2x\cos x}{(1+x^2)^2}}.$$

All four functions are differentiable for every real x; in particular, $1+x^4 > 0$ and $1+x^2 > 0$.

2. A tangent line

Let $p(x) = x\cos x$.

(a) Find $p'(x)$

$$\boxed{p'(x) = \cos x - x\sin x}.$$

(b) Tangent line at $x = \pi/2$

The point and slope are

$$p(\pi/2) = \frac{\pi}{2}\cos(\pi/2) = 0, \qquad p'(\pi/2) = 0 - \frac{\pi}{2}(1) = -\frac{\pi}{2}.$$

Hence the tangent line through $(\pi/2, 0)$ is

$$\boxed{y = -\frac{\pi}{2}\left(x - \frac{\pi}{2}\right)}.$$

3. Higher derivatives

(a) $u(x) = x^4 - 3x^2 + 1$

Differentiate successively:

$$\begin{aligned} u'(x) &= 4x^3 - 6x, \\ u''(x) &= \boxed{12x^2 - 6}, \\ u'''(x) &= \boxed{24x}, \\ u^{(4)}(x) &= 24, \\ u^{(5)}(x) &= 0. \end{aligned}$$

None of the first four derivatives is identically zero, so the smallest positive integer for which $u^{(n)}(x) = 0$ for every x is $\boxed{n = 5}$. This agrees with the fact that the $(d+1)$st derivative of a polynomial of degree d is zero.

(b) $v(x) = \cos(2x)$

Each differentiation introduces the inner derivative 2:

$$\begin{aligned} v'(x) &= -2\sin(2x), \\ v''(x) &= -2\cos(2x) \cdot 2 = \boxed{-4\cos(2x)}, \\ v'''(x) &= -4(-\sin(2x)) \cdot 2 = \boxed{8\sin(2x)}. \end{aligned}$$

4. Implicit differentiation

Consider $x^2 + 4y^2 = 20$ near $P = (2, 2)$. The point lies on the ellipse because $2^2 + 4(2^2) = 20$.

(a) Find dy/dx in terms of x and y

Treat y as a function of x and differentiate both sides:

$$2x + 8y\frac{dy}{dx} = 0.$$

Solving for the derivative gives

$$\boxed{\frac{dy}{dx} = -\frac{x}{4y}}, \qquad y \neq 0.$$

This condition holds near $P = (2, 2)$ on the branch through P.

(b) Tangent line at $P = (2, 2)$

At P, the slope is

$$\left.\frac{dy}{dx}\right|_{(2,2)} = -\frac{2}{4(2)} = -\frac{1}{4}.$$

Thus the tangent line is

$$\boxed{y - 2 = -\frac{1}{4}(x - 2)}, \qquad \text{or} \qquad \boxed{y = \frac{5}{2} - \frac{1}{4}x}.$$