Worksheet 03 Solutions

Instructor: Hong Chen **Topics:** Sections 2.6–2.8 and 3.1

1. Squeeze and important trigonometric limits

All angles are measured in radians. We use

$$\lim_{u\to 0}\frac{\sin u}{u}=1, \qquad \lim_{u\to 0}\frac{1-\cos u}{u}=0.$$

(a) $\lim_{x\to 0} x\sin(1/x^2)$

For $x\neq 0$, $|\sin(1/x^2)|\leq 1$, so

$$-|x|\leq x\sin\frac{1}{x^2}\leq |x|.$$

Both bounding functions tend to 0. By the Squeeze Theorem, the limit is $\boxed{0}$. Absolute values make these bounds valid on both sides of 0.

(b) $\lim_{x\to 0}\dfrac{\sin 5x}{\tan 2x}$

For nonzero x sufficiently close to 0,

$$\frac{\sin 5x}{\tan 2x}=\frac{5}{2}\left(\frac{\sin 5x}{5x}\right)\left(\frac{2x}{\sin 2x}\right)\cos 2x.$$

The two sine ratios and $\cos 2x$ tend to 1. Thus the limit is $\frac{5}{2}\cdot 1\cdot 1\cdot 1=\boxed{\frac{5}{2}}$.

(c) $\lim_{x\to 0}\dfrac{1-\cos 4x}{x}$

Set $u=4x$. Then $u\to 0$, and

$$\lim_{x\to 0}\frac{1-\cos 4x}{x}=4\lim_{u\to 0}\frac{1-\cos u}{u}=4\cdot 0=\boxed{0}.$$

(d) $\lim_{x\to 0}\dfrac{\sin 3x\sin 2x}{x^2}$

For nonzero x near 0,

$$\frac{\sin 3x\sin 2x}{x^2}=6\left(\frac{\sin 3x}{3x}\right)\left(\frac{\sin 2x}{2x}\right).$$

Both sine ratios tend to 1, so the limit is $6\cdot 1\cdot 1=\boxed{6}$.

2. Limits at infinity and horizontal asymptotes

(a) $R(x) = \dfrac{2x^2 - 3x + 1}{x^2 + 4}$

Divide the numerator and denominator by x^2:

$$R(x) = \frac{2 - 3/x + 1/x^2}{1 + 4/x^2}.$$

As $x \to \infty$ or $x \to -\infty$, the reciprocal powers tend to 0. Thus $\boxed{\lim_{x\to\infty} R(x) = \lim_{x\to-\infty} R(x) = 2}$, and the only horizontal asymptote is $\boxed{y = 2}$.

(b) $S(x) = \dfrac{3x}{\sqrt{x^2 + 4}}$

For $x \neq 0$, use $\sqrt{x^2} = |x|$:

$$\sqrt{x^2 + 4} = |x|\sqrt{1 + 4/x^2}, \qquad S(x) = \frac{3(x/|x|)}{\sqrt{1 + 4/x^2}}.$$

Since $x/|x| = 1$ for $x > 0$ and -1 for $x < 0$,

$$\boxed{\lim_{x\to\infty} S(x) = 3}, \qquad \boxed{\lim_{x\to-\infty} S(x) = -3}.$$

The horizontal asymptotes are $\boxed{y = 3 \text{ and } y = -3}$. Replacing $\sqrt{x^2}$ by x would give the wrong sign on the negative tail.

3. The Intermediate Value Theorem

(a) Show $p(x) = x^3 + x - 3$ has a zero in $(1, 2)$

The polynomial p is continuous on $[1, 2]$. Its endpoint values are

$$p(1) = 1 + 1 - 3 = -1, \qquad p(2) = 8 + 2 - 3 = 7.$$

Because $-1 < 0 < 7$, the IVT guarantees a number $c \in [1, 2]$ with $p(c) = 0$. Neither endpoint is a zero, so $\boxed{c \in (1, 2)}$.

(b) Does the IVT alone prove uniqueness?

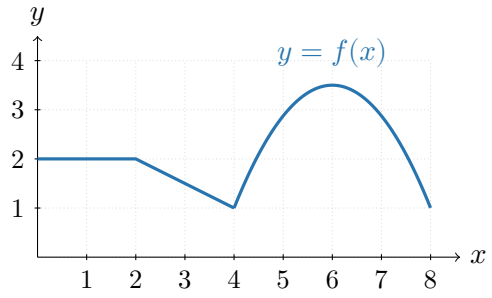
$\boxed{\text{No.}}$ The IVT proves that at least one solution exists; it does not bound the number of solutions. Proving that this particular polynomial has exactly one zero would require an additional argument, such as strict monotonicity.

(c) Why does the sign change for $q(x) = 1/x$ not imply a zero?

Although $q(-1) = -1 < 0 < 1 = q(1)$, q is undefined at 0, so it is not continuous on $[-1, 1]$. The continuity hypothesis of the IVT fails. In fact $1/x \neq 0$ for every input in its domain.

4. Derivatives from the graph

The derivative $f'(a)$ is the slope at $x = a$. The graph from the worksheet is repeated below.



(a) Find $f'(1)$, $f'(3)$, and $f'(6)$

At $x = 1$ the graph is horizontal; at $x = 3$ it lies on the line through $(2, 2)$ and $(4, 1)$; and at the smooth peak $x = 6$ the slope is 0.

$$\boxed{f'(1) = 0, \qquad f'(3) = \frac{1-2}{4-2} = -\frac{1}{2}, \qquad f'(6) = 0.}$$

(b) Where is $f'(x) < 0$, and where is $f'(x) > 0$, in the interior of the displayed domain?

The graph slopes downward on $\boxed{(2,4) \cup (6,8)}$, where $f'(x) < 0$, and upward on $\boxed{(4,6)}$, where $f'(x) > 0$. At the corners $x = 2$ and $x = 4$ the derivative does not exist, so neither point belongs to these intervals. At the peak $x = 6$, the derivative is 0.

(c) Compare $f'(4.5)$ and $f'(5.5)$

The graph rises more steeply at 4.5 than at 5.5, where it is closer to its horizontal peak. Therefore $\boxed{f'(4.5) > f'(5.5)}$.

(d) Is f differentiable at $x = 2$?

The slope on the left is 0, while the slope on the right is $-1/2$. The one-sided difference-quotient limits disagree, so $\boxed{f'(2) \text{ does not exist}}$. The graph is continuous there but has a corner.