MATH 170E Worksheet 9

Conditional distributions and expectation

Review. Let X, Y be discrete, and let B be an event with $\mathbb{P}(B) > 0$.

Conditional probabilities give a new pmf and its expectation:

$$\mathbb{P}(X = x \mid B) = \frac{\mathbb{P}(X = x, B)}{\mathbb{P}(B)}, \qquad \mathbb{E}[X \mid B] = \sum_x x\mathbb{P}(X = x \mid B).$$

For $p_Y(y) > 0$, use $\mathbb{P}(X = x \mid Y = y) = p_{X,Y}(x, y)/p_Y(y)$.

If $B_1, \ldots, B_m$ partition the sample space and each has positive probability, then

$$\mathbb{P}(X = x) = \sum_{i=1}^{m} \mathbb{P}(B_i)\,\mathbb{P}(X = x \mid B_i), \qquad \mathbb{E}[X] = \sum_{i=1}^{m} \mathbb{P}(B_i)\,\mathbb{E}[X \mid B_i].$$

Problem 1. *Roll two fair dice independently and write $S = D_1 + D_2$ for their sum. For each possible s, find the conditional pmf of D_1 given $S = s$. Use it to show that $\mathbb{E}[D_1 \mid S = s] = s/2$, and check the law of total expectation by finding $\mathbb{E}[D_1]$ from these conditional means.*

Problem 2. *Choose one of two coins with equal probability. Coin A has probability $1/2$ of heads; coin B has probability $1/4$ of heads. Use the chosen coin repeatedly, with independent tosses, and let T be the number of tosses until the first head. Find $\mathbb{P}(T = n)$ for $n \geq 1$, $\mathbb{P}(T > n)$ for $n \geq 0$, and $\mathbb{E}[T]$. For $n \geq 0$, find $\mathbb{P}(\text{coin } B \mid T > n)$.*

Problem 3. *An urn contains three red and two blue balls. Draw two balls without replacement. Let X be the number of red balls drawn, and let B be the event that the first ball is red. Find the conditional pmfs of X given B and B^c. Compute $\mathbb{E}[X \mid B]$ and $\mathbb{E}[X \mid B^c]$, and then use the law of total expectation to find $\mathbb{E}[X]$. Check your answer using indicators for the two draws.*