

Math 170E Week 7 Worksheet

Review: Markov and Chebyshev Inequalities

Let X be a random variable and let $\mu = \mathbb{E}[X]$, $\sigma^2 = \text{Var}(X)$.

- **Markov's inequality.** If $X \geq 0$ almost surely and $t > 0$, then

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}.$$

- **Chebyshev's inequality.** If $\mathbb{E}[X^2] < \infty$, then for any $t > 0$,

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}.$$

A. Deriving Chebyshev from Markov

Assume $\mathbb{E}[X^2] < \infty$.

- 1) Apply Markov's inequality to $Y := (X - \mu)^2$ to prove Chebyshev's inequality

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}, \quad t > 0.$$

- 2) Suppose $\text{Var}(X) = 1$. Give explicit numerical upper bounds from Chebyshev for $\mathbb{P}(|X - \mu| \geq 2)$ and $\mathbb{P}(|X - \mu| \geq 3)$.

B. Sharpness of Markov and Chebyshev

Let X be a random variable with $\mathbb{E}[X] = \mu$ and $X \geq 0$ a.s.

- 1) Fix $\mu > 0$ and $a > 0$. Construct an explicit distribution for a nonnegative random variable X (depending on μ and a) such that

$$\mathbb{P}(X \geq a) = \frac{\mu}{a},$$

so that Markov's inequality is *exact* for this law.

- 2) Let X be symmetric about 0, i.e. X and $-X$ have the same distribution, with $\text{Var}(X) = \sigma^2$. Use Chebyshev's inequality to give an upper bound on $\mathbb{P}(X \geq t)$ in terms of σ^2 and $t > 0$. Compare this to the trivial bound $\mathbb{P}(X \geq t) \leq \frac{1}{2}$.

C. Sample mean and weak law of large numbers via Chebyshev

Let $X_1, X_2, \dots$ be i.i.d. random variables with $\mathbb{E}[X_1] = \mu$ and $\text{Var}(X_1) = \sigma^2 < \infty$. Define the sample mean

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i.$$

- 1) Compute $\mathbb{E}[\overline{X}_n]$ and $\text{Var}(\overline{X}_n)$.
- 2) Use Chebyshev's inequality to show that, for any $\varepsilon > 0$,

$$\mathbb{P}(|\overline{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}.$$

- 3) Conclude that $\overline{X}_n \rightarrow \mu$ in probability as $n \rightarrow \infty$. *This is a standard form of the weak law of large numbers.*
- 4) For fixed $\varepsilon > 0$ and $\delta \in (0, 1)$, find a sufficient condition on n , in terms of $\sigma^2, \varepsilon, \delta$, so that

$$\mathbb{P}(|\overline{X}_n - \mu| \geq \varepsilon) \leq \delta.$$