

# Math 170E Week 6 Worksheet

## Review

- **Normal distribution.** A random variable  $X$  is  $N(\mu, \sigma^2)$  if it has density

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}, \sigma > 0.$$

The cumulative distribution function of the standard normal distribution  $N(0, 1)$  is denoted by  $\Phi(z) = \mathbb{P}(Z \leq z)$ .

- **Moment generating function (mgf).** For a real random variable  $X$ , the mgf is

$$M_X(t) = \mathbb{E}[e^{tX}], \quad t \in \mathbb{R} \text{ with } M_X(t) < \infty.$$

If it exists on a neighborhood of zero then it uniquely determines the distribution. Additionally

$$M_X^{(k)}(0) = \mathbb{E}[X^k], \quad M_{X+Y}(t) = M_X(t)M_Y(t) \text{ for independent } X, Y.$$

For  $X \sim N(\mu, \sigma^2)$ ,

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right), \quad t \in \mathbb{R}.$$

### A. Normal r.v.

Let  $X \sim N(\mu, \sigma^2)$  with  $\sigma > 0$ .

- 1) Express  $\mathbb{P}(a \leq X \leq b)$  in terms of  $\Phi$ .
- 2) For  $c > 0$ , compute  $\mathbb{P}(|X - \mu| \leq c\sigma)$  in terms of  $\Phi$ .
- 3) Let  $Y = aX + b$  with  $a \neq 0$ . Identify the distribution of  $Y$  and give its parameters.

### B. Functions of a normal variable

Let  $Z \sim N(0, 1)$ .

- 1) Define  $U = |Z|$ . Derive the density  $f_U(u)$  for  $u \geq 0$  and compute  $\mathbb{E}[U]$ .
- 2) Define  $W = Z^2$ . Find the density  $f_W(w)$  for  $w \geq 0$  and compute  $\mathbb{E}[W]$ .

### C. Sum of indep. normals is normal - proof

Let  $X_1, \dots, X_n$  be independent with  $X_i \sim N(\mu_i, \sigma_i^2)$ .

- 1) Compute the mgf  $M_S(t)$  of  $S = \sum_{i=1}^n X_i$ .
- 2) Identify the distribution of  $S$ .

**D. Linear transforms of independent normals; joint mgf**

Let  $X, Y$  be independent  $N(0, 1)$ . Define

$$U = \frac{X + Y}{\sqrt{2}}, \quad V = \frac{X - Y}{\sqrt{2}}.$$

- 1) Compute the joint mgf  $M_{U,V}(s, t) = \mathbb{E}[e^{sU+tV}]$ .
- 2) Show that  $M_{U,V}(s, t) = M_U(s) M_V(t)$ . Conclude that  $U$  and  $V$  are independent. *Hint: Approximate  $\mathbf{1}_{(-\infty, t]}(x)$  by a sequence of polynomials.*
- 3) Identify the marginal distributions of  $U$  and  $V$ .