

# Math 170E Week 5 Worksheet

## Review

- **Bernoulli distribution.** A r.v.  $X$  with support  $\{0, 1\}$  and parameter  $p \in [0, 1]$ :

$$\mathbb{P}(X = 1) = p, \quad \mathbb{P}(X = 0) = 1 - p.$$

- **Binomial distribution.**  $X \sim \text{Bin}(n, p)$  counts the number of successes in  $n$  independent Bernoulli( $p$ ) trials:

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \dots, n.$$

- **Negative binomial distribution (trials to  $r$ -th success).**  $T \sim \text{NegBin}(r, p)$  is the total number of trials needed to obtain  $r \in \mathbb{N}$  successes in i.i.d. Bernoulli( $p$ ) trials:

$$\mathbb{P}(T = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \dots$$

- **Poisson distribution.**  $X \sim \text{Pois}(\lambda)$  with rate  $\lambda > 0$ :

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

## A. Binomial tail and conditioning

Let  $X \sim \text{Bin}(n, p)$  with  $n \geq 1$ ,  $p \in (0, 1)$ .

- 1) Compute  $\frac{\mathbb{P}(X = k+1)}{\mathbb{P}(X = k)}$  for  $k = 0, \dots, n-1$ . Use the result to find the mode of  $X$ . Recall that mode is the most frequently occurring value.
- 2) For  $m \in \{0, \dots, n\}$ , compute  $\mathbb{P}(X \geq m \mid X \geq m-1)$  and simplify.
- 3) Let  $Y \sim \text{Bin}(n-1, p)$  be independent of an extra Bernoulli( $p$ ). Prove  $X \stackrel{d}{=} Y + B$ .

## B. Negative binomial identities

Let  $T \sim \text{NegBin}(r, p)$  denote trials to the  $r$ -th success.

- 1) Show  $\mathbb{P}(T > k) = \mathbb{P}(\text{Bin}(k, p) \leq r-1)$  for  $k \geq r$ .
- 2) Prove the recursion  $\mathbb{P}(T = k) = (1-p) \mathbb{P}(T = k-1) + \binom{k-2}{r-2} p^r (1-p)^{k-r}$  for  $k > r$ .
- 3) Let  $S \sim \text{NegBin}(r-1, p)$ , independent of a geometric( $p$ ) variable  $G$  (trials to first success). Show  $T \stackrel{d}{=} S + G$ .

**C. Sum and conditioning for independent Poissons (bivariate)**

Let  $X \sim \text{Pois}(\lambda_1)$  and  $Y \sim \text{Pois}(\lambda_2)$  be independent,  $\lambda_1, \lambda_2 > 0$ .

- 1) Compute the distribution of  $N := X + Y$ .
- 2) Compute the conditional distribution  $\mathbb{P}(X = k \mid N = n)$  and identify it.
- 3) Use the results above to compute  $\mathbb{P}(X \leq k \mid N = n)$  and  $\mathbb{E}[X \mid N]$ .

**D. Poisson thinning and independence (bivariate)**

Let  $N \sim \text{Pois}(\lambda)$  and, conditional on  $N$ , let  $X \mid N \sim \text{Bin}(N, p)$  with  $Y := N - X$ .

- 1) Compute the joint density  $\mathbb{P}(X = i, Y = j)$  for  $i, j \in \mathbb{N}_0$ .
- 2) Show  $X \sim \text{Pois}(\lambda p)$  and  $Y \sim \text{Pois}(\lambda(1 - p))$ .
- 3) Prove  $X$  and  $Y$  are independent. (*Hint*: factor the joint density.)

**E. Bivariate counts from  $n$  trials: dependence structure**

Perform  $n$  independent trials; on each trial outcomes are: type  $A$  w.p.  $p$ , type  $B$  w.p.  $q$ , type  $C$  w.p.  $1 - p - q$  with  $p, q \geq 0$ ,  $p + q \leq 1$ . Let  $X$  be the number of  $A$  outcomes and  $Y$  the number of  $B$  outcomes.

- 1) Compute the distribution of  $(X, Y)$ .
- 2) Find the marginals of  $X$  and  $Y$  and their expectations/variances.
- 3) Compute  $\text{Cov}(X, Y)$  and determine its sign for  $p, q > 0$ .