

Math 170E Week 4 Worksheet

Review

- **Random variable.** A random variable is a map $X : \Omega \rightarrow \mathbb{R}$. For events defined via X , we use the shorthand

$$\{X \in A\} \text{ for } X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\}.$$

- **Density function (continuous case).** A function $f : \mathbb{R}^d \rightarrow [0, \infty)$ is a probability density/mass function of an $\mathbb{R}^d$ -valued random variable X if

$$\int_{\mathbb{R}^d} f(x) dx = 1, \quad \mathbb{P}(X \in A) = \int_A f(x) dx \text{ for all } A \subseteq \mathbb{R}^d.$$

We then write $X \sim f$.

- **Expectation.** For a discrete E -valued X with density $p_X(x) = \mathbb{P}(X = x)$ and any function g ,

$$\mathbb{E}[g(X)] = \sum_{x \in E} g(x) p_X(x).$$

For a continuous, $\mathbb{R}^d$ -valued X with density f and any integrable g ,

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}^d} g(x) f(x) dx.$$

A. Discrete random variable:

Let X take values $\{0, 1, 2, 3\}$ with

$$\mathbb{P}(X = k) = c(k+1), \quad k = 0, 1, 2, 3.$$

- 1) Find the unique c for which $\mathbb{P}$ is a probability.
- 2) Compute the cumulative distribution function $F_X(t) = \mathbb{P}(X \leq t)$ for all real t .
- 3) Compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

B. Discrete random variable:

Fix $m \in \mathbb{N}$, $m \geq 2$. Let X take values $\{1, 2, \dots, m\}$ with

$$\mathbb{P}_m(X = k) = \frac{1}{kH_m}, \quad k = 1, 2, \dots, m,$$

where $H_m = \sum_{k=1}^m \frac{1}{k}$.

- 1) Compute $\mathbb{E}[X]$.
- 2) Show that $\mathbb{E}[X] \sim \frac{m}{\log m}$ as $m \rightarrow \infty$. You may use $H_m - \log m - \gamma \rightarrow 0$, where γ is the Euler-Mascheroni constant.

C. Probability density:

Define $f : \mathbb{R} \rightarrow [0, \infty)$ by

$$f(x) = \begin{cases} kx, & 0 < x < 1, \\ k(2-x), & 1 \leq x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

- 1) Determine k so that f is a density.
- 2) Compute the cumulative distribution function $F_X(t) = \int_{-\infty}^t f(x) dx$ (give a piecewise formula).
- 3) Compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

D. Exponential distribution:

Let X have density $f_\lambda(x) = \lambda e^{-\lambda x} \mathbf{1}_{\{x \geq 0\}}$ with $\lambda > 0$.

- 1) Verify that f_λ is a density.
- 2) Compute $\mathbb{E}[X]$ and $\text{Var}(X)$ as functions of λ .
- 3) Assume $\lambda \in \mathbb{N}$. For $g(x) = x^p$ with $p \in \mathbb{N}$, compute $\mathbb{E}[g(X)]$ in terms of λ and the Gamma function. Recall that

$$\begin{aligned} \Gamma(n) &= \int_0^\infty t^{n-1} e^{-t} dt \\ &= (n-1)!. \end{aligned}$$

E. Uniform Transform Theorem

Let X be a continuous real-valued random variable, and let $F_X(t) = \mathbb{P}(X \leq t)$ be its cumulative distribution function. Show that $\mathbb{P}(F_X(X) \leq t) = t$ for all $t \in [0, 1]$. Conclude that the random variable $F_X(X)$ is uniformly distributed on $[0, 1]$. Hint: How are the events $\{F_X(X) \leq t\}$ and $\{X \leq F_X^{-1}(t)\}$ related?