

Math 170E Week 3 Worksheet

Review

- **Monotonicity of probability:** If $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
- **Probability of a union:** For any events A, B it holds

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

If A and B are disjoint, meaning $A \cap B = \emptyset$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

- **Conditional probability:** If $\mathbb{P}(B) > 0$ then

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Events A, B are independent if and only if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, or equivalently $\mathbb{P}(A | B) = \mathbb{P}(A)$. Notice that an event with probability either zero or one is independent of any other event.

- **Law of total probability:** If $\{B_i\}_{i=1}^n$ is a partition of the sample space, meaning $\mathbb{P}(\bigcup_i B_i) = 1$ and $B_i \cap B_j = \emptyset$, then

$$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A | B_i) \mathbb{P}(B_i).$$

- **Bayes' theorem:** If $\mathbb{P}(A), \mathbb{P}(B) > 0$ then

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} \mathbb{P}(B | A).$$

Problems

A. Counting and independence

Let Ω be the set of all sequences of length 4 over $\{1, 2, 3, 4, 5, 6\}$.

- 1) Compute $|\Omega|$.
- 2) Let A be the event “the first coordinate is even” and B the event “the last coordinate is greater than 3.” Assume $\mathbb{P}$ is uniform. Compute $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(A \cap B)$.
- 3) Are A and B independent?

B. Law of total probability

A discrete random variable X takes values 1, 2, 3 with probabilities 1/2, 1/3, and 1/6. Given $X = i$, an event A occurs with conditional probabilities $\mathbb{P}(A | X = 1) = 1/4$, $\mathbb{P}(A | X = 2) = 1/2$, and $\mathbb{P}(A | X = 3) = 3/4$.

- 1) Compute $\mathbb{P}(A)$ using the law of total probability.
- 2) Compute $\mathbb{P}(X = 2 \mid A)$ using Bayes' theorem.

C. Conditional probability and symmetry

Two fair dice are rolled.

- 1) Let A be the event that the sum is 8 and B the event that at least one die shows a 5. Compute $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(A \mid B)$.
- 2) Are A and B independent?

D. Non-intuitive conditioning (the “Monty Hall” problem)

A prize is hidden behind one of three doors. You pick one door. The host, who knows where the prize is, opens another door revealing no prize, and offers you the chance to switch.

- 1) Compute $\mathbb{P}(\text{win if you stay})$.
- 2) Compute $\mathbb{P}(\text{win if you switch})$.

E. Bayesian updating

A test for a rare disease satisfies:

$$\mathbb{P}(\text{positive} \mid \text{disease}) = 0.99, \quad \mathbb{P}(\text{positive} \mid \text{no disease}) = 0.02,$$

and the prevalence of the disease in the population is $\mathbb{P}(\text{disease}) = 0.01$.

- 1) Compute $\mathbb{P}(\text{disease} \mid \text{positive})$.
- 2) Compute $\mathbb{P}(\text{no disease} \mid \text{positive})$.
- 3) Interpret the result in terms of the reliability of positive test results.