

Math 170E Week 2 Worksheet

A. Expectation

Let X take values $\{1, 2, 3, 4\}$ with $\mathbb{P}(X = i) = c/i^2$ for $i = 1, 2, 3, 4$.

- 1) Determine the normalizing constant c .
- 2) Compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.
- 3) Compute $\text{Var}(X)$.

B. Expectation under dependence

Let (X, Y) have joint distribution given by

$$\mathbb{P}(X = i, Y = j) = \frac{i + j}{36}, \quad i, j \in \{1, 2, 3\}.$$

- 1) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, and $\mathbb{E}[XY]$.
- 2) Are X and Y independent?
- 3) Compute $\text{Cov}(X, Y)$.

C. Conditional probability

A box contains 3 red and 2 blue balls. Two balls are drawn without replacement.

- 1) Compute $\mathbb{P}(\text{second draw is red} \mid \text{first draw is red})$.
- 2) Compute $\mathbb{P}(\text{first draw is red} \mid \text{second draw is red})$.
- 3) Verify the relationship between $\mathbb{P}(A \mid B)$ and $\mathbb{P}(B \mid A)$ using Bayes' rule.

D. Non-intuitive conditional probability

A family has two children, each equally likely to be AMAB or AFAB. Compute

$$\mathbb{P}(\text{both are AMAB} \mid \text{at least one is AMAB born on Tuesday}).$$

E. Conditional expectation (discrete)

Let (X, Y) be jointly distributed on $\{0, 1, 2\} \times \{0, 1\}$ with

$$\mathbb{P}(X = x, Y = y) = \frac{1}{9}(x + y + 1).$$

- 1) Compute $\mathbb{P}(Y = 0)$ and $\mathbb{P}(Y = 1)$.
- 2) Compute $\mathbb{E}[X \mid Y = 0]$ and $\mathbb{E}[X \mid Y = 1]$.
- 3) Compute $\mathbb{E}[X]$ and verify $\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X]$.

The formula $\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X]$ is called the *tower property* of conditional expectation, and it holds true for any random variables X, Y .