

Review

The *sample space* Ω is the set of possible outcomes. An *event* is a set $A \subseteq \Omega$. The events $A \cap B$, $A \cup B$, and $A^c = \Omega \setminus A$ mean “both,” “at least one,” and “not A ,” respectively.

For a finite sample space, probability assigns a number $\mathbb{P}(A)$ to each event. The basic rules are

$$\mathbb{P}(A) \geq 0, \quad \mathbb{P}(\Omega) = 1, \quad \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \quad \text{if } A \cap B = \emptyset.$$

In particular, $\Omega = A \dot{\cup} A^c$ (a disjoint union), so $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$. Also, the two disjoint decompositions $A \cup B = A \dot{\cup} (B \setminus A)$ and $B = (B \setminus A) \dot{\cup} (A \cap B)$ give

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

If all outcomes are *equally likely*, then $\mathbb{P}(A) = |A|/|\Omega|$. Check this assumption before counting outcomes. In each problem, write down the relevant events and justify the rule you use.

Problems

1. A fair die

Roll one fair six-sided die. Let A be the event that the result is odd, and B the event that the result is at least 4.

- (a) Write Ω , A , B , A^c , $A \cup B$, and $A \cap B$ as sets.
- (b) Calculate $\mathbb{P}(A)$, $\mathbb{P}(B)$, $\mathbb{P}(A \cup B)$, and $\mathbb{P}(A \cap B)$. For each answer, show which outcomes you counted.
- (c) Prove $(A \cup B)^c = A^c \cap B^c$ by showing that an arbitrary $\omega \in \Omega$ belongs to the left side exactly when it belongs to the right side. Check the identity against your sets from (a).

2. Two coins

Flip two *independent*, fair coins, labeled first and second. Let H be the event “exactly one head” and F the event “first coin is heads.”

- (a) List the equally likely outcomes in Ω . Write H , F , and $H \cap F$ as subsets of Ω , then compute their probabilities.
- (b) Suppose you record only the *number* of heads. Its possible values are 0, 1, 2. Give the probability of each value, and explain why the three values cannot be treated as equally likely.

3. Basic consequences of the rules

Let A and B be events. Prove each claim using disjoint sets; write down the decomposition you use.

- (a) $\mathbb{P}(\emptyset) = 0$. *Hint:* $\Omega = \Omega \dot{\cup} \emptyset$.
- (b) If $A \subseteq B$, then $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$. Deduce $\mathbb{P}(A) \leq \mathbb{P}(B)$.

4. A loaded die

The die has outcomes $1, \dots, 6$, with $\mathbb{P}(\{k\}) = ck$ for a constant $c > 0$.

- (a) Determine c from the axioms. Why do these six probabilities define a probability on all subsets of Ω ?
- (b) Find the probability of an even outcome. Explain why counting three even faces out of six would give the wrong answer.

5. Probabilities of overlapping events

Suppose $\mathbb{P}(A^c) = 1/3$, $\mathbb{P}(B) = 1/2$, and $\mathbb{P}(A \cap B) = 1/4$. Find $\mathbb{P}(A \cup B)$, $\mathbb{P}(A \setminus B)$, and the probability that neither event occurs. Show how each event is obtained from A and B .

6. Two dice

Roll two independent fair dice. Take the ordered pair of results as the outcome. Let A be the event “the sum is 7” and B the event “at least one die shows 6.” Write Ω , A , B , and $A \cap B$ as sets or set-builder descriptions. Find $\mathbb{P}(A \cup B)$, showing where you account for outcomes counted twice. Explain why the eleven possible sums $2, 3, \dots, 12$ are not equally likely.