

Fall 2024 MATH33A Worksheet 3: Sections 2.3, 2.4, 3.1

Exercise 1. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$.

(a) Compute A^{-1} .

(b) Use the inverse to find all solutions to $A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, and all solutions to $A\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

(a) $A^{-1} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & 4 & -6 \\ 1 & -1 & 2 \end{bmatrix}$.

(b) $A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ implies $\vec{x} = A^{-1} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$,

$$\vec{x} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & 4 & -6 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ -29 \\ 9 \end{bmatrix}$$

Thus the only solution is $\vec{x} = \begin{bmatrix} 15 \\ -29 \\ 9 \end{bmatrix}$.

Similarly for the second equation, we find that

$$\vec{x} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & 4 & -6 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

is the only solution.

Exercise 2. Show that the following subsets are *not* subspaces of $\mathbb{R}^2$:

(a) $V = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$

(b) $V = \left\{ \begin{bmatrix} 3s+1 \\ 2-s \end{bmatrix} \mid s \in \mathbb{R} \right\}$

Show that the following subsets *are* subspaces of $\mathbb{R}^2$:

(c) $V = \left\{ \begin{bmatrix} t \\ 3s \end{bmatrix} \mid s, t \in \mathbb{R} \right\}.$

(d) $V = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

To show a subset is *not* a subspace, we need to find $u, v \in V$ such that $u + v \notin V$ (i.e., V is not *closed under addition*), or some $u \in V$ and a scalar $c \in \mathbb{R}$ such that $cu \notin V$ (i.e., V is not *closed under scalar multiplication*).

(a) Let $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, which are both in V . Then $u + v = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is not an element of V , so V is not a subspace.

(b) Let $u = \begin{bmatrix} 3+1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 7 \\ 0 \end{bmatrix} \in V$, so $u + v = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$. Let us show that $u + v$ is not in V , so there does not exist $s \in \mathbb{R}$ such that $\begin{bmatrix} 3s+1 \\ 2-s \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$. In other words, we aim to show that there are no solutions to the linear system

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} [s] = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

Thus performing augmented row reduction, we have

$$\left(\begin{array}{c|c} 3 & 10 \\ -1 & -1 \end{array} \right) \Rightarrow \text{swap and multiply by } -1 \left(\begin{array}{c|c} 1 & 1 \\ 3 & 10 \end{array} \right) \Rightarrow \left(\begin{array}{c|c} 1 & 1 \\ 0 & 7 \end{array} \right)$$

Thus, there are no solutions to $\begin{bmatrix} 3 \\ -1 \end{bmatrix} [s] = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$, so $\begin{bmatrix} 11 \\ 1 \end{bmatrix}$ is not in V , so V is not a subspace.

To show subsets are subspaces of $\mathbb{R}^2$, we have to show that for **all** $u, v \in V$ that $u + v$ is in V (V is *closed under addition*), and that for all $v \in V, c \in \mathbb{R}$, that $c \cdot v \in V$ (V is *closed under scalar multiplication*).

(c) Let u, v be two elements of V , so $u = \begin{bmatrix} t_1 \\ 3s_1 \end{bmatrix}$ for some $t_1, s_1 \in \mathbb{R}$ and $v = \begin{bmatrix} t_2 \\ 3s_2 \end{bmatrix}$ for some $t_2, s_2 \in \mathbb{R}$.

Then, $u + v = \begin{bmatrix} t_1 + t_2 \\ 3s_1 + 3s_2 \end{bmatrix}$. In particular, $u + v = \begin{bmatrix} t \\ 3s \end{bmatrix}$ for $t = t_1 + t_2, s = s_1 + s_2$, so $u + v \in V$.

Now let u be an element of V , so $u = \begin{bmatrix} t_1 \\ 3s_1 \end{bmatrix}$ for some $t_1, s_1 \in \mathbb{R}$, and let $c \in \mathbb{R}$ be arbitrary. Then

$$c \cdot u = \begin{bmatrix} ct_1 \\ 3cs_1 \end{bmatrix} = \begin{bmatrix} t \\ 3s \end{bmatrix} \text{ for } t = ct_1, s = 3s_1, \text{ so } c \cdot u \in V.$$

(d) Let u, v be two elements of V . Since V only has a single vector, we must have $u = v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then $u + v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$, so V is closed under addition. Now let $u \in V$, so $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and let $c \in \mathbb{R}$ arbitrary. Then $c \cdot u = \begin{bmatrix} c \cdot 0 \\ c \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$, so V is closed under scalar multiplication.

Exercise 3. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation of projection onto the line $y = x$. Is T invertible? Argue both (1) geometrically, and (2) algebraically by finding the matrix representation for T and computing its determinant.

Geometrically: Since T is projection onto the line $y = x$, for every point (a, b) on the line $y = -x$ perpendicular to $y = x$, $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, T cannot have an inverse since it is not one-to-one/injective.

Algebraically: Since T is projection onto $y = x$, for every point (a, b) on the line $y = x$, $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. Similarly for every point (a, b) on the line $y = -x$ perpendicular to $y = x$, $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Using linearity of T , we have:

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = \frac{1}{2} T \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Similarly,

$$T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = T \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = \frac{1}{2} T \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Therefore, T is represented by the matrix $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. Since $\det A = 0$, T is not invertible.

Exercise 4. Let A be an $m \times n$ matrix, and let B be an **invertible** $n \times n$ matrix.

- (a) Suppose that for all $\vec{b} \in \mathbb{R}^m$, $Ax = \vec{b}$ has a solution. What does this tell you about the image of A ?
 - (b) How many solutions are there to $Bx = \vec{b}$ for any $\vec{b} \in \mathbb{R}^n$?
 - (c) What is the image of B (hint: does $Bx = \vec{b}$ always have a solution?)
 - (d) **(Challenge)** If B is invertible, what is $\text{Im}(AB)$ in terms of the images of B, A ? If C is an invertible $m \times m$ matrix, can you answer the same question for $\text{Im}(CA)$?
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- (a) Let $\vec{b} \in \mathbb{R}^m$ be any vector. Since $Ax = \vec{b}$ has a solution, $\vec{b}$ is in the image of A . Therefore, $\text{Im}A$ is all of $\mathbb{R}^m$ since it contains every element of $\mathbb{R}^m$.
- (b) There is *exactly one* solution since B is invertible, given by $x = B^{-1}\vec{b}$.
- (c) By part (a), (b), $\text{Im}B = \mathbb{R}^n$.
- (d) $\text{Im}(AB)$ is the set of all the vectors in $\mathbb{R}^m$ of the form ABx for $x \in \mathbb{R}^n$. $\text{Im}(A)$ is the set of all the vectors in $\mathbb{R}^m$ of the form Ay for $y \in \mathbb{R}^n$. Therefore, every element of $\text{Im}(AB)$ is also an element of $\text{Im}(A)$ since $ABx = Ay$ for $y = Bx \in \mathbb{R}^n$, i.e., $\text{Im}(AB)$ is a subset of $\text{Im}(A)$. Now take any element Ay of $\text{Im}(A)$. Since B is invertible, there is a solution x to $Bx = y$. Thus, $Ay = ABx$, so $Ay \in \text{Im}(AB)$. Therefore, $\text{Im}(A)$ is a subset of $\text{Im}(AB)$, so since both sets are subsets of the other they are equal: $\text{Im}(A) = \text{Im}(AB)$.

There is no characterization of $\text{Im}(CA)$ in terms of $\text{Im}(A)$. For instance, let $A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then if

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\text{Im}(CA) = \text{Im}(A)$ is the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. But if $C = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (which is

invertible), then $\text{Im}(CA)$ is the span of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so the image depends on the choice of C .

Exercise 5. Find the inverse of the following matrix

$$A = \begin{bmatrix} 1 & c & c^3 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

in terms of $c \in \mathbb{R}$. Verify your answer with matrix multiplication.

$$A^{-1} = \begin{bmatrix} 1 & -c & c^2 - c^3 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

Fall 2024 MATH 33A Worksheet 5: 3.3, 3.4

Exercise 1. Let $A : \mathbb{R}^8 \rightarrow \mathbb{R}^7$ be given by the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 & 2 \\ 2 & 3 & 0 & -1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Determine $\dim(\ker A)$ (hint: use rank-nullity and find $\dim(\operatorname{im} A)$.)

$\operatorname{im} A$ is two dimensional by observation or by noticing that the row reduced echelon form has exactly two leading ones. Therefore, since $\dim \ker A + \dim \operatorname{im} A = 8$ and $\dim \operatorname{im} A = 2$, we have $\dim \ker A = 6$.

Exercise 2. Determine whether the vector $\vec{x}$ is in the span V of the vectors $\vec{v}_1, \vec{v}_2$. If it is, express $\vec{x}$ as a linear combination of $\vec{v}_1, \vec{v}_2$.

(a) $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

(b) $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(a) $\vec{x}$ is not a linear combination of $\vec{v}_1, \vec{v}_2$ because $\vec{x}$ is not contained in

$$\operatorname{span}(\vec{v}_1, \vec{v}_2) = \operatorname{span}(\vec{v}_1) = \{k\vec{v}_1, k \in \mathbb{R}\} = \left\{ \begin{bmatrix} k \\ 2k \end{bmatrix}, k \in \mathbb{R} \right\}.$$

(b) Let $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, then $S^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$, and so

$$[\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x} = S^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Exercise 3.

(a) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. What matrix S would satisfy $B = S^{-1}AS$?

- (b) Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Recall that $[\vec{x}]_{\mathfrak{B}}$ is the $\mathfrak{B}$ -coordinate vector of $\vec{x}$. Are $A\vec{x}$ and $SB[\vec{x}]_{\mathfrak{B}}$ the same vectors? Justify your answer.
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- (a) Let $S = [v_1 \ v_2] = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, then $S^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$, and so

$$B = S^{-1}AS = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -12 \\ 4 & 10 \end{bmatrix}.$$

- (b) • The long way: By the previous exercise, we know that

$$[\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x} = S^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

and so

$$SB[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -5 & -12 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -7 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

On the other hand,

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

Therefore, $A\vec{x}$ and $SB[\vec{x}]_{\mathfrak{B}}$ are the same vectors.

- The short way:

$$A\vec{x} = T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}} = SB[\vec{x}]_{\mathfrak{B}}$$

Exercise 4. True or false: Explain your reasoning or find an example or counterexample.

- (a) If V is a subspace of $\mathbb{R}^3$ that does not contain any of the elementary column vectors e_1, e_2, e_3 , then $V = \{\vec{0}\}$.
- (b) If v_1, v_2, v_3, v_4 are linearly independent vectors, then v_1, v_2, v_3 are linearly independent.
- (c) If v_1, v_2, v_3 are linearly independent vectors, then v_1, v_2, v_3, v_4 are linearly independent.
- (d) There exists a 4×4 matrix A with $\ker A = \text{span}(e_1, e_2, e_3)$ and $\text{im } A = \text{span}(e_3 + e_4)$.
- (e) There exists a 5×5 matrix A with $\ker A = \text{im } A$.
- (f) There exists a 4×4 matrix A with $\ker A = \text{im } A$.
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- (a) This is false. For instance, $V = \text{span}\left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\rangle$ is not the zero subspace but doesn't contain any of e_1, e_2, e_3 .
- (b) This is true. Since v_1, v_2, v_3, v_4 are linearly independent, the only solution (x_1, x_2, x_3, x_4) to $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = \vec{0}$ is $(0, 0, 0, 0)$. Therefore, the only solution to $x_1v_1 + x_2v_2 + x_3v_3 = \vec{0}$ is $(0, 0, 0)$, so v_1, v_2, v_3 are linearly independent.

- (c) This is false. For instance, let $v_1 = e_1, v_2 = e_2, v_3 = e_3$ in $\mathbb{R}^3$, and let v_4 be any vector in $\mathbb{R}^3$.
- (d) Yes. Notice that in this case $\dim \ker A + \dim \operatorname{im} A = 3 + 1 = 4$, so there is no issue from rank nullity. Since $e_1, e_2, e_3 \in \ker A$, the first three columns of A must be zero, and so we find the following example:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- (e) False. A 5×5 matrix must have $\dim \ker A + \dim \operatorname{im} A = 5$ by rank-nullity, and since 5 is not an even number, we cannot have $\dim \ker A = \dim \operatorname{im} A$.
- (f) True. Notice that by rank nullity, we must have $\dim \ker A = \dim \operatorname{im} A = 2$, so suppose A is a 4×4 matrix with $\ker A = \operatorname{span}\langle e_1, e_2 \rangle$. This implies its first two columns are the zero vector. In order for the image of A to also be $\operatorname{span}\langle e_1, e_2 \rangle$, the last two columns must span $\operatorname{span}\langle e_1, e_2 \rangle$, so the following matrix works:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 5. Consider two subspaces V and W of $\mathbb{R}^n$, where V is contained in W , denoted $V \subseteq W$.

- (a) Show that $\dim(V) \leq \dim(W)$.
- (b) Show that if $\dim(V) = \dim(W)$, then $V = W$.
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- (a) Let $\mathfrak{B} = \{v_1, v_2, \dots, v_k\}$ be a basis for V . Since $V \subset W$, each of the vectors $v_1, v_2, \dots, v_k$ are in W , and thus $\mathfrak{B}$ is a linearly independent subset of W . Therefore, we can choose vectors $v_{k+1}, \dots, v_l \in W$ such that $\{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_l\}$ forms a basis for W , so $\dim W = l \geq k = \dim V$.
- (b) Let $\mathfrak{B} = \{v_1, v_2, \dots, v_k\}$ be a basis for V and since $V \subset W$, each of the vectors $v_1, \dots, v_k$ are in W , so $\mathfrak{B}$ is a linearly independent subset of W . Since $\dim W = \dim V = k$ by assumption, $\mathfrak{B}$ is thus a basis for W (see Theorem 3.3.4 in Bretscher).

Fall 2024 MATH 33A Worksheet 5: 5.3, 5.4

Exercise 1. True or false:

- (a) If A is orthogonal then it is invertible.
- (b) If A is symmetric it is invertible.
- (c) The entries of an orthogonal matrix are all less than or equal to 1 in absolute value.
- (d) Let V be a subspace of $\mathbb{R}^n$ and B the matrix for orthogonal projection onto V . Then $B^2 = B$.
- (e) If $\vec{v}_1, \dots, \vec{v}_m$ is a basis of unit length vectors for a subspace V , there is an orthonormal basis of V containing the vectors $\vec{v}_1$ and $\vec{v}_2$.
- (f) For all $\vec{v}, \vec{w} \in \mathbb{R}^n$, $|\vec{v} \cdot \vec{w}|^2 \leq \|\vec{v}\|^2 \|\vec{w}\|^2$ with equality if and only if $\vec{v}, \vec{w}$ are perpendicular.

Solution

- (a) True. Since A is orthogonal, A^{-1} exists and is equal to A^T .
- (b) False. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric but not invertible.
- (c) True. Let $A = [u_1 \ \dots \ u_n]$ be orthogonal, so $\{u_1, \dots, u_n\}$ are orthonormal. Therefore for each of the u_i with entries $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have $1 = \|u_i\| = \sqrt{x_1^2 + \dots + x_n^2}$, so $x_j^2 \leq 1$ for all j , so all the elements of A are bounded by 1 in absolute value.
- (d) True. For any $v \in \mathbb{R}^n$, $B^2v = B(Bv)$. Since $\text{im } B = V$ and B fixes V since B is orthogonal projection on to V , $B(Bv) = Bv$. Therefore, since $B^2v = Bv$ for all $v \in \mathbb{R}^n$, $B^2 = B$.
- (e) False. Consider

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

which are unit length and form a basis for $\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle$. Since $v_1 \cdot v_2 \neq 0$, we cannot form an orthonormal basis with v_1, v_2 .

- (f) False. If “perpendicular” was replaced with “parallel”, this is true and is the Cauchy Schwarz inequality. But if v, w are perpendicular, then $v \cdot w = 0$ and thus $v \cdot w \neq \|v\|^2\|w\|^2$ for any non-zero perpendicular v, w .

Exercise 2. Let

$$A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 2 & 1 & 1 & -4 \end{bmatrix}$$

- (a) Find an orthonormal basis $\mathfrak{B} = \{\vec{u}_1, \vec{u}_2\}$ for $\ker A$.
 (b) Using your basis from part (a), find the matrix B for orthogonal projection onto $\ker A$.
 (c) Extend $\mathfrak{B}$ to an orthonormal basis $\mathfrak{E}$ for $\mathbb{R}^4$. Find the matrix for the orthogonal projection in part (b) with respect to the basis $\mathfrak{E}$.

Solution

- (a) RREF: $\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 2 \end{bmatrix}$, which results in $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ being a basis for $\ker A$. Performing

Gram-Schmidt with $v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, we have $v_1^\perp = v_1$ and

$$v_2^\perp = v_2 - \frac{(v_1^\perp \cdot v_2)}{\|v_1^\perp\|^2} v_1^\perp = v_2 + \frac{5}{3} v_1^\perp = \begin{bmatrix} 4/3 \\ -1/3 \\ 5/3 \\ 1 \end{bmatrix}$$

Thus, we get

$$\vec{u}_1 = v_1^\perp / \|v_1^\perp\| = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, \vec{u}_2 = v_2^\perp / \|v_2^\perp\| = \begin{bmatrix} 4/\sqrt{51} \\ -1/\sqrt{51} \\ 5/\sqrt{51} \\ 3/\sqrt{51} \end{bmatrix}$$

- (b) Let $Q = [\vec{u}_1 \quad \vec{u}_2]$. Then $B = QQ^T$, so

$$B = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 4/\sqrt{51} \\ \frac{1}{\sqrt{3}} & -1/\sqrt{51} \\ \frac{1}{\sqrt{3}} & 5/\sqrt{51} \\ 0 & 3/\sqrt{51} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 4/\sqrt{51} & -1/\sqrt{51} & 5/\sqrt{51} & 3/\sqrt{51} \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 11 & -7 & 1 & 4 \\ -3 & 6 & 4 & -1 \\ -7 & 4 & 14 & 5 \\ 4 & -1 & 5 & 3 \end{bmatrix}$$

- (c) Let e_1, e_2, e_3, e_4 be the standard basis in $\mathbb{R}^4$. Note that v_1, v_2, e_1, e_2 are linearly independent, so they form a basis for $\mathbb{R}^4$. Applying Gram-Schmidt to this basis, we obtain an orthonormal basis $\mathfrak{E} = \{u_1, u_2, u_3, u_4\}$ that extends $\mathfrak{B}$, where

$$u_3 = \begin{bmatrix} \frac{\sqrt{102}}{17} \\ \frac{1}{\sqrt{102}} \\ -\frac{1}{\sqrt{102}} \\ -\frac{2\sqrt{102}}{51} \end{bmatrix}, \quad u_4 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{\sqrt{6}}{3} \end{bmatrix}.$$

Notice that $B\vec{u}_1 = \vec{u}_1$ since $\vec{u}_1 \in \ker A$, and similarly $B\vec{u}_1 = \vec{u}_1$. Since u_3, u_4 are perpendicular to $\ker A$, $Bu_3 = Bu_4 = 0$. Therefore

$$[\text{proj}_{\ker A}]_{\mathfrak{E}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 3. Find all orthogonal 2×2 matrices.

Solution The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an orthogonal 2×2 matrix if and only if the following conditions are met: $a^2 + c^2 = 1$, $b^2 + d^2 = 1$, $ab + cd = 0$. The first two conditions imply that there exists angles θ, ω so that $a = \cos \theta, c = \sin \theta, b = \sin \omega, d = \cos \omega$. The last condition implies that

$$0 = \cos \theta \sin \omega + \sin \theta \cos \omega = \sin(\theta + \omega).$$

We may pick $\theta + \omega = 0$ or $\theta + \omega = \pi$. Thus, all 2×2 orthogonal matrices are of the form

$$A = \begin{bmatrix} \cos \theta & \sin(-\theta) \\ \sin \theta & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

or

$$A = \begin{bmatrix} \cos \theta & \sin(\pi - \theta) \\ \sin \theta & \cos(\pi - \theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$$

where $\theta \in [0, 2\pi]$.

Exercise 4. Let $A = \begin{bmatrix} 3 & 2 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & 4 & -6 & -3 \end{bmatrix}$, $V = \text{im} A$.

(a) Find the projection matrix B for proj_V , projection onto V .

(b) Using B , determine whether $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \in \text{im} A$ (B is projection onto V , so $\vec{v}$ is in V if and only if $B \cdot \vec{v} = \vec{v}$).

(c) Find the least squares solution to $A\vec{x} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$.

Solution

(a) First, we find a basis for V . An RREF of A is

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so we see that

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

form a basis for $V = \text{im } A$. Applying Gram-Schmidt to $\vec{v}_1, \vec{v}_2$, we obtain

$$\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - \sqrt{10} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{11}} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$

Then $\vec{u}_1, \vec{u}_2$ form an orthonormal basis for V . Let $Q = [\vec{u}_1 \quad \vec{u}_2]$. Then

$$B = QQ^T = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{11} \\ 0 & -1/\sqrt{11} \\ 1/\sqrt{10} & 3/\sqrt{11} \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & 0 & 1/\sqrt{10} \\ -1/\sqrt{11} & -1/\sqrt{11} & 3/\sqrt{11} \end{bmatrix} = \begin{bmatrix} 109/110 & 1/11 & 3/110 \\ 1/11 & 1/11 & -3/11 \\ 3/110 & -3/11 & 101/110 \end{bmatrix}.$$

(b) We compute that

$$B\vec{v} = \frac{1}{110} \begin{bmatrix} 319 \\ -2 \\ 473 \end{bmatrix} \neq \vec{v},$$

so $\vec{v} \notin V = \text{im } A$.

(c) We find

$$A^T A = \begin{bmatrix} 10 & 10 & 0 & 0 \\ 10 & 21 & -22 & -11 \\ 0 & -22 & 44 & 22 \\ 0 & -11 & 22 & 11 \end{bmatrix}$$

and

$$A^T \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 24 \\ -22 \\ -11 \end{bmatrix}.$$

A solution to the system $A^T A \vec{x} = A^T \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ is $\begin{bmatrix} 3/10 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. This is one, but not all least square solutions.

Exercise 5. Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution First we perform Gram Schmidt on the columns resulting in the vectors

$$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

Thus, $Q = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]$. Notice that R is a matrix which satisfies $A = QR$, so $Q^{-1}A = R$. Since Q is orthogonal we have $Q^{-1} = Q^T$, so $R = Q^T A$. Thus, we have

$$R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & 2 & 1 \\ 0 & 0 & \frac{3}{\sqrt{2}} \end{bmatrix}$$

Fall 2024 MATH 33A Worksheet 8: 6.1, 6.2, 6.3

Exercise 1. Compute the determinant of the following matrices:

(a) $\begin{bmatrix} 3 & -5 \\ 4 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & -5 & 1 \\ 3 & 0 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 2 & 5 \\ 1 & -1 & -2 \\ 6 & -2 & -3 \end{bmatrix}$

Solution

(a) 17

(b) 8

(c) -69

(d) -2

Exercise 2. Determine whether the following are true or false:

(a) $\det(A + B) = \det(A) + \det(B)$ for any two $n \times n$ matrices A and B .

(b) If B is the rref of A , then $\det(B) = \det(A)$.

(c) There exists a 3×3 matrix A with real valued entries such that $A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(d) If A is an orthogonal matrix, then $\det(A) = \pm 1$.

Solution

(a) False. Here's a counter-example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad \det \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) + \det \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) = 1 + 1 \neq \det \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = 0$$

(b) False, scaling a matrix or swapping rows changes the determinant!

(c) False. Such a matrix A would have $\det(A^2) = -1 \implies \det(A) = \sqrt{-1}$, however, if A is composed of all real entries, this cannot be the case.

(d) True. If A is orthogonal, we know that $A^T = A^{-1}$ and thus we have that $\det(A) = \det(A^T) = \det(A^{-1}) = \frac{1}{\det(A)} \implies \det(A)^2 = 1 \implies \det(A) = \pm 1$

Exercise 3. Write down the relationship between $\det(B)$ and $\det(A)$ for the following scenarios
(**Challenge:** Can you prove any or all of them?)

- (a) If B is obtained by multiplying a row of A by a scalar k .
- (b) If B is obtained by swapping two rows of A .
- (c) If B is obtained by adding a multiple of one row of A to another row of A .
- (d) $B = A^T$
- (e) If A is invertible, $B = A^{-1}$

Solution

- (a) $\det(B) = k \det(A)$
- (b) $\det(B) = -\det(A)$
- (c) $\det(B) = \det(A)$
- (d) $\det(B) = \det(A)$
- (e) $\det(B) = \frac{1}{\det(A)}$

Exercise 4. Consider the linear system given by

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

- (a) Use Cramer's Rule to solve this system.
- (b) Find the inverse of the above matrix in terms of its determinant and "classical adjoint" (adjugate). Use this inverse to solve the system.

Solution

- (a) The determinant of the matrix is

$$2 \cdot \det \begin{bmatrix} 4 & 5 \\ 0 & 7 \end{bmatrix} - 3 \cdot \det \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 0 & 4 \\ 6 & 0 \end{bmatrix} = 146$$

The components of $\vec{x}$ are

$$x_1 = \frac{1}{146} \det \begin{bmatrix} 5 & 3 & 0 \\ 6 & 4 & 5 \\ 7 & 0 & 7 \end{bmatrix} = \frac{119}{146}, \quad x_2 = \frac{1}{146} \det \begin{bmatrix} 2 & 5 & 0 \\ 0 & 6 & 5 \\ 6 & 7 & 7 \end{bmatrix} = \frac{164}{146}, \quad x_3 = \frac{1}{146} \det \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 6 & 0 & 7 \end{bmatrix} = \frac{44}{146}$$

- (b) The determinant of the matrix is

$$2 \cdot \det \begin{bmatrix} 4 & 5 \\ 0 & 7 \end{bmatrix} - 3 \cdot \det \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 0 & 4 \\ 6 & 0 \end{bmatrix} = 146$$

The classical adjoint is

$$\text{adj}(A) = \begin{bmatrix} 28 & -21 & 15 \\ 30 & 14 & -10 \\ -24 & 18 & 8 \end{bmatrix}$$

Therefore the inverse is

$$A^{-1} = \frac{1}{146} \begin{bmatrix} 28 & -21 & 15 \\ 30 & 14 & -10 \\ -24 & 18 & 8 \end{bmatrix}$$

and the solution to the system is

$$A^{-1}\vec{b} = \frac{1}{146} \begin{bmatrix} 28 & -21 & 15 \\ 30 & 14 & -10 \\ -24 & 18 & 8 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = \frac{1}{146} \begin{bmatrix} 119 \\ 164 \\ 44 \end{bmatrix}$$

Fall 2024 MATH 33A Worksheet 10: 7.1, 7.2, 7.3

Exercise 1. True or false:

- (a) The eigenvalues of a 2×2 matrix A are the solutions of the equation $\lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = 0$.
- (b) If a real matrix A has only the eigenvalues 1 and -1 , then A must be orthogonal.
- (c) If A and B are two 3×3 matrices such that $\operatorname{tr} A = \operatorname{tr} B$ and $\det A = \det B$, then A and B must have the same eigenvalues.
- (d) If A is a noninvertible $n \times n$ matrix, then the geometric multiplicity of eigenvalue 0 is $n - \operatorname{rank}(A)$.
- (e) If matrix A^2 is diagonalizable over $\mathbb{R}$, then matrix A must be diagonalizable over $\mathbb{R}$ as well.
- (f) If $\vec{v}$ and $\vec{w}$ are linearly independent eigenvectors of matrix A , then $\vec{v} + \vec{w}$ must be an eigenvector of A as well.

Solution

- (a) True. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then its characteristic equation is

$$(a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - (\operatorname{tr} A)\lambda + (\det A),$$

so the eigenvalues of A are exactly the solutions of this equation.

- (b) False. Having special eigenvalues do not imply orthogonality. Pick any two non-orthogonal vectors $v_1, v_2 \in \mathbb{R}^2$, say $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Consider T that maps $T(v_1) = v_1, T(v_2) = -v_2$. Then its matrix is $A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$. It has eigenvalues 1 and -1 , but it is not orthogonal since its column vectors are not orthonormal.
- (c) False. The trace and the determinant determines 2 coefficients of the characteristic polynomial of any given matrix. The characteristic polynomial of a 3×3 matrix has 3 free coefficients, so there's still 1 free coefficient. Thus, this does not ensure that different matrices have the same eigenvalues. As a counterexample, we can take

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If A, B are two 2×2 matrices, this will actually be true.

- (d) True. The eigenspace E_0 corresponding to eigenvalue 0 is exactly the kernel $\ker A$, and we have by rank-nullity theorem,

$$\dim(E_0) = \dim(\ker A) = n - \text{rank}(A).$$

- (e) False. If λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2 . If we can find A so that A^2 only has one distinct eigenvalue -1 , which does not have any real root, then we can be sure that A can not be diagonalizable. For example, consider the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Then $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is a diagonal matrix, hence diagonalizable. However, the characteristic polynomial of A is $\lambda^2 + 1$. This equation does not have real roots, so the matrix A is not diagonalizable over $\mathbb{R}$.

- (f) False. Consider $A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$. It has eigenvalues 1 and -1 , where $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is its eigenvector with eigenvalue 1 and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is its eigenvector with eigenvalue -1 . $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. However, $\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not an eigenvector of A since

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

is not a multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Exercise 2. What is the characteristic polynomial of

$$A = \begin{bmatrix} 1 & a & b \\ 0 & -1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

and what are its eigenvalues? For which values of a, b, c is the matrix diagonalizable?

Solution The characteristic polynomial is

$$f_A(\lambda) = \det \begin{bmatrix} 1 - \lambda & a & b \\ 0 & -1 - \lambda & c \\ 0 & 0 & 1 - \lambda \end{bmatrix} = (1 - \lambda)(-1 - \lambda)(1 - \lambda) = -(\lambda - 1)^2(\lambda + 1)$$

This equation has roots $\lambda = \pm 1$, meaning $-1, 1$ are the eigenvalues of the matrix A . To find the eigenvectors, we need to solve equations $A\vec{x} = \vec{x}$ and $A\vec{x} = -\vec{x}$. These have solutions

$$\text{Eigenvalue 1: } \left[\begin{array}{ccc|c} 1-1 & a & b & 0 \\ 0 & -1-1 & c & 0 \\ 0 & 0 & 1-1 & 0 \end{array} \right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \text{ if } ac+2b \neq 0, \text{ else } \left[\begin{array}{ccc|c} 0 & 1 & -c/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{Eigenvalue } -1: \left[\begin{array}{ccc|c} 1+1 & a & b & 0 \\ 0 & -1+1 & c & 0 \\ 0 & 0 & 1+1 & 0 \end{array} \right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & a/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

If $ac + 2b \neq 0$, the eigenspaces are $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ for eigenvalue 1 and $\text{span}\left\{\begin{bmatrix} -\frac{a}{2} \\ 1 \\ 0 \end{bmatrix}\right\}$ for eigenvalue -1 .

Since the dimension of these two subspaces together is only 2, A is not diagonalizable in this case.

If $ac + 2b = 0$ the eigenspaces are $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ c/2 \\ 1 \end{bmatrix}\right\}$ and $\text{span}\left\{\begin{bmatrix} -\frac{a}{2} \\ 1 \\ 0 \end{bmatrix}\right\}$ for eigenvalue -1 . In this case A is diagonalizable.

Exercise 3. Starting from $F_0 = 0, F_1 = 1$, the n -th Fibonacci number is defined iteratively by $F_{n+1} = F_n + F_{n-1}$. This gives us a sequence

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, \dots$$

- (a) Let $x_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ for $n = 0, 1, 2, \dots$. Find a 2×2 matrix A so that $x_{n+1} = Ax_n$.
- (b) Find the characteristic polynomial and the eigenvalues of A .
- (c) Find the eigenvectors of A .
- (d) Diagonalize A .
- (e) (Challenging) Show that

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

Solution

- (a) We have

$$x_{n+1} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x_n.$$

Thus $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

- (b) The characteristic polynomial of A is

$$\det(A - \lambda I_2) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1.$$

The eigenvalues of A are roots of the characteristic polynomials, so they are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

- (c) The eigenspaces of A are

$$E_{\lambda_1} = \ker(A - \lambda_1 I_2) = \ker \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} = \ker \begin{bmatrix} 1 & \frac{-1-\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix}$$

$$E_{\lambda_2} = \ker(A - \lambda_2 I_2) = \ker \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix} = \ker \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix}$$

We can choose eigenvectors $v_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$.

(d) Let $S = [v_1, v_2] = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$, then by the inverse matrix formula, we get

$$S^{-1} = \frac{1}{\det A} \operatorname{adj}(A) = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}.$$

Then

$$D = S^{-1}AS$$

$$\text{where } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}, S = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}, S^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}.$$

(e) We have

$$x_n = Ax_{n-1} = A^2x_{n-2} = \dots = A^n x_0$$

where $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We deduce that

$$x_n = A^n x_0 = S \underbrace{(S^{-1}AS)(S^{-1}AS) \dots (S^{-1}AS)}_{n \text{ times}} S^{-1} x_0 = SD^n S^{-1} x_0$$

where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Thus

$$x_n = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \left(\frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix}.$$

Recall that $x_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$, we conclude that

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$