

Math 32A — Week 9 Worksheet

Tangent Plane, Linear Approximation, Gradient, Directional Derivative

Review

Gradient. For $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, the gradient is

$$\nabla f(x, y) = \begin{pmatrix} f_x(x, y) \\ f_y(x, y) \end{pmatrix}.$$

For $F : \mathbb{R}^3 \rightarrow \mathbb{R}$, the gradient is

$$\nabla F(x, y, z) = \begin{pmatrix} F_x(x, y, z) \\ F_y(x, y, z) \\ F_z(x, y, z) \end{pmatrix}.$$

Directional derivative. If u is a unit vector, then

$$D_u f(a, b) = \nabla f(a, b) \cdot u.$$

The maximum directional derivative at (a, b) equals $\|\nabla f(a, b)\|$, and it occurs in the direction of $\nabla f(a, b)$ if $\nabla f(a, b) \neq 0$.

Tangent plane and linear approximation. If $z = f(x, y)$, the tangent plane at $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The linear approximation of f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Tangent plane to a level surface. For a level surface $\{(x, y, z) : F(x, y, z) = \alpha\}$ and a point (a, b, c) on it, the tangent plane is

$$\left\{ (x, y, z) \in \mathbb{R}^3 : \nabla F(a, b, c) \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) = 0 \right\}.$$

1. **Gradient and directional derivatives in $\mathbb{R}^2$.** Let

$$f(x, y) = xe^y + y^2.$$

- (a) Compute $\nabla f(x, y)$.
- (b) Evaluate $\nabla f(1, 0)$.
- (c) Let $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Find the unit vector u in the direction of v , then compute $D_u f(1, 0)$.
- (d) Find the maximum rate of increase of f at $(1, 0)$ and a unit direction where it occurs.

2. Tangent plane and numerical approximation. Let

$$f(x, y) = \ln(9 + x^2 + 4y).$$

- (a) Compute $f_x(x, y)$ and $f_y(x, y)$.
- (b) Find the tangent plane to $z = f(x, y)$ at $(0, 0, f(0, 0))$.
- (c) Use the linear approximation at $(0, 0)$ to estimate $f(0.10, -0.02)$.

3. Tangent line to a level curve. Let

$$f(x, y) = x^2 + xy + y^2.$$

Consider the level curve $f(x, y) = 7$.

- (a) Verify that $(2, 1)$ lies on the level curve.
- (b) Compute $\nabla f(2, 1)$.
- (c) Use $\nabla f(2, 1)$ as a normal vector to find an equation of the tangent line to the level curve at $(2, 1)$.

4. Tangent plane to a level surface in $\mathbb{R}^3$. Let

$$F(x, y, z) = x^2 + y^2 + z^2 - 2x - 4y + z.$$

Let $P = (1, 2, 0)$.

- (a) Compute $\nabla F(x, y, z)$ and evaluate $\nabla F(P)$.
- (b) Compute $F(P)$.
- (c) Find the tangent plane to the level surface $F(x, y, z) = F(P)$ at the point P .

5. Chain rule via gradients along a path. Let

$$T(x, y) = x^2y + \sin(xy), \quad \mathbf{r}(t) = \begin{pmatrix} t \\ 1 + t^2 \end{pmatrix}.$$

Define $S(t) = T(\mathbf{r}(t))$.

- (a) Compute $\nabla T(x, y)$.
- (b) Compute $\mathbf{r}'(t)$ and evaluate $\mathbf{r}'(0)$.
- (c) Compute $S'(0)$ using

$$S'(t) = \nabla T(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

- (d) Interpret $S'(0)$ as a directional derivative of T at the point $\mathbf{r}(0)$. State the direction as a unit vector.