

Math 32A — Week 8 Worksheet

Partial Derivatives for Functions $\mathbb{R}^2 \rightarrow \mathbb{R}$ and $\mathbb{R}^3 \rightarrow \mathbb{R}$

Review

Partial derivatives. If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, the partial derivatives at (a, b) are

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h},$$

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b + h) - f(a, b)}{h},$$

when these limits exist. Higher partial derivatives include f_{xx} , f_{yy} , and mixed partials f_{xy} , f_{yx} .

Schwarz's Theorem. If the mixed partials f_{xy} and f_{yx} are continuous near a point (a, b) , then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Gradient and directional derivatives. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable, its gradient is

$$\nabla f = \begin{pmatrix} f_{x_1} \\ \vdots \\ f_{x_n} \end{pmatrix}.$$

For a unit vector u , the directional derivative is

$$\frac{\partial}{\partial u} f(p) = \nabla f(p) \cdot u.$$

Tangent plane. For a surface $z = f(x, y)$, the tangent plane at $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

1. **Compute first partial derivatives.** Let

$$f(x, y) = x^2y + e^{xy} + 3 \sin(x).$$

- (a) Compute $f_x(x, y)$ and $f_y(x, y)$.
- (b) Evaluate $f_x(0, 2)$ and $f_y(0, 2)$.

2. **Partial derivatives from the definition.** Let

$$g(x, y) = \sqrt{x^2 + 4y^2}.$$

- (a) Compute $g_x(0, 0)$ and $g_y(0, 0)$ using the limit definition.
- (b) Compute $g_x(x, y)$ and $g_y(x, y)$ for $(x, y) \neq (0, 0)$.
- (c) Is g differentiable at $(0, 0)$. Briefly justify your answer.

3. Second partial derivatives and mixed partials. Let

$$h(x, y) = x^3 y^2 - 2x \cos(y).$$

- (a) Compute $h_{xx}(x, y)$ and $h_{yy}(x, y)$.
- (b) Compute $h_{xy}(x, y)$ and $h_{yx}(x, y)$.
- (c) Verify that $h_{xy} = h_{yx}$.

4. Tangent plane and a linear approximation. Let

$$f(x, y) = \ln(4 + x^2 + y^2).$$

- (a) Compute $f_x(x, y)$ and $f_y(x, y)$.
- (b) Find the equation of the tangent plane to $z = f(x, y)$ at the point with $x = 1$ and $y = 1$.
- (c) Use the tangent plane to estimate $f(1.02, 0.97)$.

5. A field in $\mathbb{R}^3$: rates of change. A temperature field is given by

$$T(x, y, z) = x^2 y + y z^2 + e^{x-z}.$$

- (a) Compute $T_x(x, y, z)$, $T_y(x, y, z)$, and $T_z(x, y, z)$.
- (b) Evaluate the gradient $\nabla T(1, 2, 0)$.
- (c) Find the directional derivative at $(1, 2, 0)$ in the direction of the vector $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.
- (d) At $(1, 2, 0)$, in what direction does T increase most rapidly, and what is the maximum rate of increase.