

Math 32A — Midterm Preparation Worksheet

Vectors, Curves, Vector-Valued Functions, Limits

Review

Dot product. For $u, v \in \mathbb{R}^n$, $\cos \theta = \frac{u \cdot v}{\|u\| \|v\|}$ when $u \neq 0$ and $v \neq 0$.

Cross product. For $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$,

$$u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}, \quad \|u \times v\| = \text{area of the parallelogram spanned by } u, v.$$

Planes. A plane with normal $n \neq 0$ through $P_0 = (x_0, y_0, z_0)$ satisfies

$$n \cdot ((x, y, z) - (x_0, y_0, z_0)) = 0.$$

Parametric curves. If $(x(t), y(t))$ is a curve and $x'(t) \neq 0$, then

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$

Limits in $\mathbb{R}^2$. To prove a limit exists at $(0, 0)$, polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ are often useful. To show a limit does not exist, compare two different paths.

1. **Vectors, area, and a plane.** Let $A = (1, 0, 2)$, $B = (3, 1, 1)$, $C = (2, -1, 4)$.

- (a) Compute $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in column form.
- (b) Compute $\overrightarrow{AB} \cdot \overrightarrow{AC}$ and the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
- (c) Compute $\overrightarrow{AB} \times \overrightarrow{AC}$ and the area of triangle $\triangle ABC$.
- (d) Find an equation of the plane through A, B, C .

2. **Line-plane intersection and angle.** Let the line be

$$\mathbf{r}(t) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix},$$

and let the plane be

$$2x - y + 2z = 5.$$

- (a) Find the intersection point of the line and plane, or explain why it does not exist.
- (b) Let v be the direction vector of the line and n be a normal vector to the plane. Compute $v \cdot n$.

- (c) Let ϕ be the angle between the line and the plane. Use

$$\sin \phi = \frac{|v \cdot n|}{\|v\| \|n\|}$$

to compute ϕ .

3. **Parametric curve: tangents and special points.** Consider the plane curve

$$x(t) = t^2 - 1, \quad y(t) = t^3 - 3t.$$

- (a) Compute $\frac{dy}{dx}$ as a function of t where it is defined.
- (b) Find all values of t for which the tangent line is horizontal.
- (c) Determine whether the curve has a vertical tangent. If yes, find the corresponding t and the point on the curve.

4. **Vector-valued functions and speed.** Let

$$\mathbf{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}.$$

- (a) Compute the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ and acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$.
- (b) Compute the speed $\|\mathbf{v}(t)\|$ and show that it is constant.
- (c) Find the unit tangent vector $\mathbf{T}(t)$.
- (d) Find an equation of the tangent line at $t = 0$.

5. **Limits with parameters and continuity at $(0,0)$.** Fix real constants a and b . Define

$$F(x, y) = \begin{cases} \frac{ax^2 + bxy + ay^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ k, & (x, y) = (0, 0). \end{cases}$$

- (a) Convert $F(x, y)$ to polar coordinates $(x, y) = (r \cos \theta, r \sin \theta)$ for $(x, y) \neq (0, 0)$, and simplify the expression so it has no r in the denominator.
- (b) For which values of b does the limit $\lim_{(x,y) \rightarrow (0,0)} F(x, y)$ exist. State the limit in terms of a .
- (c) Choose k so that F is continuous at $(0, 0)$ whenever the limit exists.
- (d) When the limit does not exist, prove this by comparing the limits along the two paths $y = x$ and $y = -x$.