

Math 32A — Week 6 Worksheet

Functions of Several Variables: Limits and Continuity

Review

Functions of several variables. A function of two variables is a map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, written $f(x, y)$. Its graph is the surface in $\mathbb{R}^3$ given by $z = f(x, y)$. Level sets are curves $f(x, y) = c$ in the xy -plane.

Limits. We say $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y) - L| < \varepsilon.$$

If different paths approaching (a, b) give different limiting values, then the limit does not exist.

Continuity. A function f is continuous at (a, b) if

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b).$$

Algebraic combinations and compositions of continuous functions are continuous on their domains.

Polar coordinates near the origin. Using $x = r \cos \theta$, $y = r \sin \theta$, we have $(x, y) \rightarrow (0, 0)$ if and only if $r \rightarrow 0$. This is often useful to prove limits.

1. **Domain, range, and level sets.** Define

$$f(x, y) = \sqrt{4 - x^2 - y^2}.$$

- (a) Find the domain of f .
- (b) Find the range of f .
- (c) Describe the graph $z = f(x, y)$ geometrically.
- (d) For a constant c with $0 \leq c \leq 2$, describe the level set $f(x, y) = c$.

2. **A limit that does not exist.** Consider

$$g(x, y) = \frac{x^2 y}{x^4 + y^2}.$$

- (a) Compute $\lim_{(x,y) \rightarrow (0,0)} g(x, y)$ along the path $y = 0$.
- (b) Compute $\lim_{(x,y) \rightarrow (0,0)} g(x, y)$ along the path $y = x^2$.
- (c) Conclude whether $\lim_{(x,y) \rightarrow (0,0)} g(x, y)$ exists.

3. **A limit that exists (use polar coordinates).** Define

$$h(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Show that $\lim_{(x,y) \rightarrow (0,0)} h(x, y) = 0$. Your proof should use polar coordinates or a direct inequality.

4. **Continuity of a piecewise function.** Let

$$p(x, y) = \begin{cases} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

- (a) Determine whether p is continuous at $(0, 0)$.
- (b) Determine whether p is continuous at every point $(a, b) \neq (0, 0)$.

5. **Will the asteroid hit Los Angeles?** Work in units where positions are vectors in $\mathbb{R}^3$ and time is $t \geq 0$. The center of the Earth follows

$$E(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix}.$$

The position of Los Angeles relative to the Earth's center is the vector

$$\ell = \begin{pmatrix} \cos(\pi t) \\ \sin(\pi t) \\ 1 \end{pmatrix}.$$

The asteroid follows the path

$$A(t) = \begin{pmatrix} (t-3)^2 \\ t \\ 11-t \end{pmatrix}.$$

Define the position of Los Angeles as $L(t) = E(t) + \ell$ and the distance from the asteroid to Los Angeles as

$$d(t) = \|A(t) - L(t)\|.$$

Assume the asteroid “hits Los Angeles” if $d(t) \leq 0.2$ for some $t \geq 0$.

- (a) Write $L(t)$ explicitly and simplify $A(t) - L(t)$.
- (b) Compute an explicit formula for $d(t)$.
- (c) Explain briefly why $d(t)$ attains a minimum.
- (d) Find the minimum value of $d(t)$ and decide whether the asteroid hits Los Angeles under the rule above.