

Math 32A — Week 4 Worksheet

Vector-Valued Functions and Calculus on Them

Review

A three-dimensional **vector-valued function** is a map $f : \{1, 2, 3\} \rightarrow \mathbb{R}^n$ written

$$f(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Derivatives are taken componentwise:

$$f'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}.$$

For motion, velocity is $\mathbf{v}(t) = f'(t)$ and acceleration is $\mathbf{a}(t) = f''(t)$. Speed is

$$\|\mathbf{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.$$

The unit tangent vector is

$$\mathbf{T}(t) = \frac{f'(t)}{\|f'(t)\|} \quad (\text{when } f'(t) \neq 0).$$

Differentiation rules (when defined):

$$\begin{aligned} \frac{d}{dt}(f(t) \cdot \mathbf{s}(t)) &= f'(t) \cdot \mathbf{s}(t) + f(t) \cdot \mathbf{s}'(t), \\ \frac{d}{dt}(f(t) \times \mathbf{s}(t)) &= f'(t) \times \mathbf{s}(t) + f(t) \times \mathbf{s}'(t), \\ \frac{d}{dt}\|f(t)\| &= \frac{f(t) \cdot f'(t)}{\|f(t)\|} \quad (\text{when } f(t) \neq 0). \end{aligned}$$

1. **Differentiate and evaluate.** Let

$$f(t) = \begin{pmatrix} t^2 \\ \sin t \\ e^t \end{pmatrix}.$$

- (a) Compute $f'(t)$ and $f''(t)$.
- (b) Evaluate $f(0)$, $f'(0)$, and $f''(0)$.

2. **Tangent line to a space curve.** Consider

$$f(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}.$$

- (a) Find the point $f(\frac{\pi}{2})$ and the velocity $f'(\frac{\pi}{2})$.
- (b) Write a parametric equation for the tangent line at $t = \frac{\pi}{2}$.

3. **Speed and arc length.** Let

$$f(t) = \begin{pmatrix} t \\ t^2 \\ 0 \end{pmatrix}, \quad 0 \leq t \leq 1.$$

- (a) Compute $f'(t)$ and the speed $\|f'(t)\|$.
- (b) Write an integral for the arc length of the curve on $[0, 1]$.
- (c) Compute the unit tangent vector $\mathbf{T}(t)$.

4. **Dot product rule and orthogonality.** Let

$$f(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \quad \mathbf{v}(t) = f'(t).$$

- (a) Compute $\mathbf{v}(t)$.
- (b) Compute $\frac{d}{dt}(f(t) \cdot f(t))$ in two ways:
 - (i) by simplifying $f(t) \cdot f(t)$ first,
 - (ii) by using the dot product differentiation rule.
- (c) Use your result to show that $f(t) \cdot \mathbf{v}(t) = 0$ for all t .

5. **Cross product rule and a conserved quantity.** Let

$$f(t) = \begin{pmatrix} t \\ t^2 \\ 1 \end{pmatrix}, \quad \mathbf{s}(t) = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}.$$

- (a) Compute $f'(t)$ and $\mathbf{s}'(t)$.
- (b) Use the cross product differentiation rule to compute $\frac{d}{dt}(f(t) \times \mathbf{s}(t))$.
- (c) Evaluate $f(0) \times \mathbf{s}(0)$ and $\frac{d}{dt}(f(t) \times \mathbf{s}(t))|_{t=0}$.