

Math 32A — Week 3 Worksheet

Parametric Curves in $\mathbb{R}^2$

Review

A **parametric curve** in $\mathbb{R}^2$ is given by functions $x(t)$, $y(t)$ on an interval $I \subseteq \mathbb{R}$:

$$\gamma(t) = (x(t), y(t)), \quad t \in I.$$

The **velocity** is $\gamma'(t) = (x'(t), y'(t))$, and the **speed** is

$$\|\gamma'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

The **length** of the curve is

$$l(\gamma) = \int_I \|\gamma'(t)\| dt.$$

If $x'(t) \neq 0$, then the slope of the tangent line satisfies

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$

If $x'(t) \neq 0$, a second-derivative formula is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

1. **Eliminate the parameter.** Consider the curve

$$x(t) = 2 \cos t, \quad y(t) = 3 \sin t, \quad 0 \leq t \leq 2\pi.$$

(a) Eliminate t , i.e. represent the curve in the form

$$\{(x, y) \in \mathbb{R}^2 : \dots\},$$

where $\dots$ should be replaced by a condition that x, y must satisfy for the point (x, y) to lie on the curve.

(b) Describe the curve geometrically.

(c) Indicate the orientation (clockwise or counterclockwise) as t increases.

2. **Tangent line from $\frac{dy}{dx}$.** Let

$$x(t) = t^2 - 1, \quad y(t) = t^3 - 3t.$$

(a) Compute $\frac{dy}{dx}$ as a function of t .

(b) Find the equation of the tangent line at $t = 1$.

- (c) Find all t for which the tangent line is horizontal.

3. Speed and distance traveled. A particle moves along

$$\gamma(t) = (t, t^2), \quad 0 \leq t \leq 2.$$

- (a) Compute the velocity $\gamma'(t)$ and the speed $\|\gamma'(t)\|$.
(b) Write an integral expression for the total distance traveled on $0 \leq t \leq 2$.

4. Cusps and self-intersections. Consider

$$x(t) = t^2, \quad y(t) = t^3 - 3t.$$

- (a) Find all t where $\gamma'(t) = (0, 0)$.
(b) Find all t where $x'(t) = 0$ (vertical tangent candidates). Determine whether the tangent is vertical there.
(c) Determine whether the curve has a self-intersection, i.e. distinct $t_1 \neq t_2$ with $\gamma(t_1) = \gamma(t_2)$. If so, find the intersection point.

5. Curve length Let

$$x(t) = \cos t, \quad y(t) = \sin t, \quad 0 < t < \pi.$$

- (a) Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of t .
(b) Find the length of the parametrized curve.