

Math 32A — Week 2 Worksheet

Dot Product, Cross Product, Planes

Review

Dot product If $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$, then

$$u \cdot v = u_1 v_1 + \dots + u_n v_n, \quad \|u\| = \sqrt{u \cdot u}.$$

If $u \neq 0$ and $v \neq 0$, the angle $\theta \in [0, \pi]$ between them satisfies

$$\cos \theta = \frac{u \cdot v}{\|u\| \|v\|}.$$

The projection of v onto u is

$$\text{proj}_u(v) = \frac{v \cdot u}{u \cdot u} u \quad (u \neq 0).$$

Cross product (only in $\mathbb{R}^3$). For $u, v \in \mathbb{R}^3$, $u \times v$ is perpendicular to both u and v , and

$$\|u \times v\| = \|u\| \|v\| \sin \theta,$$

so $\|u \times v\|$ is the area of the parallelogram spanned by u and v . The scalar triple product $u \cdot (v \times w)$ gives the signed volume of the parallelepiped spanned by u, v, w ; the (unsigned) volume is $|u \cdot (v \times w)|$. The formula for the cross product is

$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}.$$

1. **Dot product and angle.** Let

$$u = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

- (a) Compute $u \cdot v$, $\|u\|$, and $\|v\|$.
- (b) Find $\cos \theta$, where θ is the angle between u and v .
- (c) Decide whether the angle is acute, right, or obtuse.

2. **Projection and decomposition.** Let

$$u = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}.$$

- (a) Compute $\text{proj}_u(v)$.
- (b) Write $v = v_{\parallel} + v_{\perp}$ where $v_{\parallel}$ is parallel to u and $v_{\perp}$ is orthogonal to u .

- (c) Check that $u \cdot v_{\perp} = 0$.

3. Cross product and area. Let

$$u = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

- (a) Compute $u \times v$.
(b) Find the area of the parallelogram spanned by u and v .
(c) Find the area of the triangle with sides u and v (sharing a vertex).

4. Scalar triple product and volume. Let

$$u = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

- (a) Compute $v \times w$.
(b) Compute $u \cdot (v \times w)$.
(c) Find the volume of the parallelepiped spanned by u, v, w .

5. Planes and intersections.

- (a) Find an equation of the plane through $P_0 = (1, -1, 2)$ with normal vector $n = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

- (b) Find an equation of the plane through the three points

$$A = (1, 0, 0), \quad B = (0, 1, 0), \quad C = (0, 0, 1).$$

- (c) Find the point where the line $r(t) = (1, 2, 3) + t(1, -1, 2)$ intersects the plane from part (a), if it intersects.