

Math 32A — Week 10 Worksheet

Multivariate Chain Rule and Optimization

Review

Chain rule with a path. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable and $r : \mathbb{R} \rightarrow \mathbb{R}^n$ is differentiable, then for $S(t) = f(r(t))$,

$$S'(t) = \nabla f(r(t)) \cdot r'(t).$$

Chain rule with intermediate variables. If $z = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$, then $z = z(u, v)$ and

$$\begin{aligned}\frac{\partial z}{\partial u} &= f_x(x(u, v), y(u, v)) \frac{\partial x}{\partial u} + f_y(x(u, v), y(u, v)) \frac{\partial y}{\partial u}, \\ \frac{\partial z}{\partial v} &= f_x(x(u, v), y(u, v)) \frac{\partial x}{\partial v} + f_y(x(u, v), y(u, v)) \frac{\partial y}{\partial v}.\end{aligned}$$

Unconstrained optimization. For $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, critical points satisfy $\nabla f = 0$. The Hessian test uses

$$D = f_{xx}f_{yy} - (f_{xy})^2.$$

If $D > 0$ and $f_{xx} > 0$, there is a local minimum. If $D > 0$ and $f_{xx} < 0$, there is a local maximum. If $D < 0$, there is a saddle point.

Constrained optimization. To optimize $f(x, y)$ subject to $g(x, y) = c$, use Lagrange multipliers:

$$\nabla f = \lambda \nabla g, \quad g(x, y) = c.$$

1. **Chain rule along a curve.** Let

$$f(x, y) = x^2y + e^{xy}, \quad r(t) = \begin{pmatrix} \cos t \\ 1 + t \end{pmatrix}.$$

Define $S(t) = f(r(t))$.

- (a) Compute $\nabla f(x, y)$.
- (b) Compute $S'(t)$ using $S'(t) = \nabla f(r(t)) \cdot r'(t)$.
- (c) Evaluate $S'(0)$.

2. **Chain rule with intermediate variables.** Let

$$z = f(x, y) = \ln(x^2 + y^2), \quad x = u^2 - v, \quad y = e^{uv}.$$

- (a) Compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ in terms of u and v .
- (b) Evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at $(u, v) = (1, 0)$.

3. Find and classify critical points. Let

$$f(x, y) = x^3 - 3x + y^2 - 4y.$$

- (a) Find all critical points.
- (b) Classify each critical point as a local maximum, local minimum, or saddle point.
- (c) Determine whether f has an absolute minimum on $\mathbb{R}^2$. Briefly justify.

4. Optimization with a geometric constraint. Find the maximum and minimum values of

$$f(x, y) = x + 2y$$

subject to the constraint

$$x^2 + y^2 = 25.$$

- (a) Use Lagrange multipliers to find all constrained critical points.
- (b) State the maximum and minimum values and where they occur.
- (c) Interpret your answer geometrically using the gradient.

5. A constrained design problem. A closed cylindrical can must have volume $V = 100\pi$. Let r be the radius and h be the height. The surface area is

$$A(r, h) = 2\pi r^2 + 2\pi rh.$$

- (a) Write the constraint equation relating r and h .
- (b) Use the constraint to eliminate h and write A as a function of r only.
- (c) Find the value of r that minimizes A and then find the corresponding h .