

MATH 171 Worksheet 8

Renewal processes and one harder Poisson process problem

Review. Let $T_1, T_2, \dots$ be iid positive random variables with $0 < \mathbb{E}[T_1] < \infty$. Define renewal times

$$S_n = T_1 + \dots + T_n, \quad N(t) = \max \{n : S_n \leq t\}.$$

Then

$$\frac{N(t)}{t} \rightarrow \frac{1}{\mathbb{E}[T_1]} \quad \text{as } t \rightarrow \infty.$$

If each renewal cycle also has reward R_i , where the pairs (T_i, R_i) are iid and $\mathbb{E}[|R_1|] < \infty$, then

$$\frac{R_1 + \dots + R_{N(t)}}{t} \rightarrow \frac{\mathbb{E}[R_1]}{\mathbb{E}[T_1]}.$$

This is the renewal reward principle.

In an alternating renewal process, each cycle consists of time W_i in state 1 and time U_i in state 2. If

$$\mathbb{E}[W_1] = \mu_W, \quad \mathbb{E}[U_1] = \mu_U,$$

then the long-run fraction of time spent in state 1 is

$$\frac{\mu_W}{\mu_W + \mu_U}.$$

For independent Poisson processes with rates $\lambda_1, \dots, \lambda_k$, the combined process is Poisson with rate $\lambda_1 + \dots + \lambda_k$. Conditional on an arrival in the combined process, its type is i with probability

$$\frac{\lambda_i}{\lambda_1 + \dots + \lambda_k}.$$

Problem 1. A laboratory instrument alternates between working and repair. Its working periods are iid with mean 18 days, and its repair periods are iid with mean 2 days.

Find the long-run fraction of time that the instrument is working. Also find the long-run rate, per day, at which repairs are completed.

Problem 2. A company replaces a delivery van at renewal times. The lifetime of one van has mean 6 years. During its lifetime, the van costs 1200 dollars per year to maintain, and replacement costs 24000 dollars.

Find the long-run average cost per year.

Problem 3. Two independent Poisson streams arrive at a server. Type A arrivals have rate 2, and type B arrivals have rate 3. The server stops observing when either 4 type A arrivals have occurred or 3 type B arrivals have occurred.

Find the probability that the type A threshold is reached first. Then express the expected stopping time as a finite sum.