

MATH 171 Worksheet 7

Poisson processes, waiting times, and random sums

Review. A random variable T has exponential distribution with rate $\lambda > 0$ if

$$\mathbb{P}(T > t) = e^{-\lambda t}, \quad t \geq 0.$$

Then

$$\mathbb{E}[T] = \frac{1}{\lambda}, \quad \text{Var}(T) = \frac{1}{\lambda^2}.$$

The exponential distribution is memoryless:

$$\mathbb{P}(T > s + t \mid T > s) = \mathbb{P}(T > t).$$

A Poisson process $\{N(t)\}_{t \geq 0}$ with rate λ satisfies

$$N(t) \sim \text{Poisson}(\lambda t), \quad \mathbb{P}(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

It has independent increments, and

$$N(t + s) - N(s) \sim \text{Poisson}(\lambda t).$$

Equivalently, the interarrival times are independent exponential random variables with rate λ . In particular,

$$\mathbb{P}(N(t) = 0) = e^{-\lambda t}.$$

If $0 < s < t$, then conditioning on the total number of arrivals gives

$$\mathbb{P}(N(s) = m \mid N(t) = n) = \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}.$$

For a compound Poisson sum

$$S(t) = Y_1 + \cdots + Y_{N(t)},$$

where the Y_i are iid and independent of $N(t)$,

$$\mathbb{E}[S(t)] = \lambda t \mathbb{E}[Y_1], \quad \text{Var}(S(t)) = \lambda t \mathbb{E}[Y_1^2].$$

Problem 1. Emails arrive at a help desk according to a Poisson process with rate 4 per hour.

Find the probability that exactly 3 emails arrive in the first hour and exactly 2 emails arrive in the next half hour. Then, given that 6 emails arrived in the first two hours, find the probability that exactly 4 of them arrived in the first hour.

Problem 2. Cars pass a crossing according to a Poisson process with rate 12 cars per minute. A pedestrian needs a gap of length 5 seconds with no cars to cross safely.

Suppose the pedestrian starts looking immediately after a car passes. Find the probability that the first gap is safe. Then find the expected number of unsafe gaps before the first safe gap.

Problem 3. Insurance claims arrive according to a Poisson process with rate 3 per week. Claim sizes are iid, independent of the arrival process, with mean 8 thousand dollars and standard deviation 5 thousand dollars.

Find the mean and standard deviation of the total amount paid over 4 weeks.