

MATH 171 Worksheet 6

Exit distributions, exit times, and first-step equations

Review. Let $\{X_n\}_{n \geq 0}$ be a Markov chain with transition matrix P . For a set A of states, define the hitting time

$$V_A := \min \{n \geq 0 : X_n \in A\}.$$

For a single state a , write V_a instead of $V_{\{a\}}$.

- For disjoint target sets A and B , the exit probability

$$h(x) := \mathbb{P}_x(V_A < V_B)$$

satisfies the boundary conditions

$$h(x) = 1 \quad \text{for } x \in A, \quad h(x) = 0 \quad \text{for } x \in B,$$

and the first-step equation

$$h(x) = \sum_y p(x, y)h(y)$$

for all other states x .

- For a target set A , the expected hitting time

$$g(x) := \mathbb{E}_x[V_A]$$

satisfies

$$g(x) = 0 \quad \text{for } x \in A,$$

and the first-step equation

$$g(x) = 1 + \sum_y p(x, y)g(y)$$

for all states $x \notin A$.

- These equations come from conditioning on the first step. In finite-state problems where the target set is eventually reached with probability one, the relevant solution is unique.

Problem 1. A student enters a two-year program as a freshman. At the end of each academic year, a freshman remains a freshman with probability 0.20, becomes a sophomore with probability 0.65, and leaves the program with probability 0.15. A sophomore remains a sophomore with probability 0.25, graduates with probability 0.60, and leaves the program with probability 0.15. Graduation and leaving the program are absorbing outcomes.

Let the states be F, S, G, L , where F means freshman, S means sophomore, G means graduated, and L means left the program. Starting from F , find the probability that the student eventually graduates. Also find the expected number of years until the student either graduates or leaves the program.

Problem 2. A research project is reviewed after each month. Its progress score is modeled by a Markov chain X_n on

$$\{0, 1, 2, 3, 4, 5\}.$$

State 0 means the project is cancelled, and state 5 means the project is funded. Both are absorbing. For $1 \leq x \leq 4$, the score increases to $x + 1$ with probability $3/5$ and decreases to $x - 1$ with probability $2/5$.

Starting from $X_0 = 2$, find the probability that the project is funded before it is cancelled. Then find the expected number of monthly reviews until one of the two absorbing outcomes occurs.

Problem 3. A factory tests one batch each day. Each batch is defective with probability $1/5$, independently of all other days. The factory starts a full inspection as soon as it observes two defective batches in a row.

Define a Markov chain whose state records enough information to decide how close the factory is to inspection. Starting with no previous defective batch, compute the expected number of days until inspection begins.