

MATH 171 Worksheet 5

Random walks on finite graphs, limit behavior, and ergodicity

Review. Let $G = (V, E)$ be a finite graph. For $x \in V$, write $\deg(x)$ for the number of neighbors of x .

- The **simple random walk** on G is the Markov chain with transition probabilities

$$p(x, y) = \begin{cases} \frac{1}{\deg(x)} & \text{if } x \sim y \\ 0 & \text{if } x \not\sim y \end{cases}.$$

- A probability vector π is **stationary** if

$$\pi P = \pi.$$

For simple random walk on a finite connected graph,

$$\pi(x) = \frac{\deg(x)}{\sum_{v \in V} \deg(v)}$$

is stationary.

- A finite-state chain is **irreducible** if every state can be reached from every other state. It is **aperiodic** if returns to a state are not forced to occur only at multiples of some integer greater than 1.
- If a finite-state chain is irreducible and aperiodic, then

$$p^n(x, y) \rightarrow \pi(y) \quad \text{as } n \rightarrow \infty.$$

Periodicity can prevent this pointwise limit from existing.

- For an irreducible finite-state chain, long-run averages are governed by π . In particular, for a function f on the state space,

$$\frac{1}{n} \sum_{k=1}^n f(X_k) \rightarrow \sum_x f(x) \pi(x).$$

This is the ergodic-average interpretation of the stationary distribution. The convergence holds with probability one.

Problem 1. A student wanders among five campus locations

$$V = \{A, B, C, D, E\},$$

where A is an apartment, B is Bruin Walk, C is a classroom, D is a dining hall, and E is the library. The walkways are

$$AB, \quad AC, \quad BC, \quad BD, \quad CE.$$

At each step, the student chooses uniformly among the available walkways and moves to the neighboring location at the other end.

Find the stationary distribution of this walk. Use it to decide which location is visited most often in the long run and which location is visited least often. Also compute the long-run fraction of time spent on Bruin Walk.

Problem 2. A robot patrols four checkpoints arranged in a square:

$$1 \sim 2, \quad 2 \sim 3, \quad 3 \sim 4, \quad 4 \sim 1.$$

At each step it moves uniformly to one of the two neighboring checkpoints.

Starting from checkpoint 1, determine whether $p^n(1, 1)$ converges as $n \rightarrow \infty$. Then modify the patrol rule as follows: at each step the robot stays put with probability $1/2$, and with probability $1/2$ it moves uniformly to one of its two neighbors. For the modified chain, find the limiting distribution of the robot's location.

Problem 3. Use the same campus graph as in Problem 1. A security guard follows the simple random walk on this graph during a long patrol. Each visit has inspection cost

$$c(A) = 2, \quad c(B) = 5, \quad c(C) = 4, \quad c(D) = 1, \quad c(E) = 3.$$

Compute the long-run average inspection cost per step. Explain why, for a very long patrol, the answer is independent of the guard's starting location.