

MATH 171 Worksheet 4

Stationary distributions and detailed balance

Review. Let $\{X_n\}_{n \geq 0}$ be a Markov chain with transition matrix $P = (p(x, y))$.

- A probability vector π is a **stationary distribution** if

$$\pi P = \pi.$$

Equivalently,

$$\pi(\{y\}) = \sum_x \pi(\{x\})p(x, y) \quad \text{for each state } y.$$

- A nonnegative function π is a **stationary measure** if $\pi P = \pi$, but one does *not* require

$$\sum_x \pi(\{x\}) = 1.$$

- The **detailed balance condition** is

$$\pi(\{x\})p(x, y) = \pi(\{y\})p(y, x) \quad \text{for all states } x, y.$$

If a probability vector π satisfies detailed balance, then π is stationary.

- For a birth-death chain, detailed balance gives the recursion

$$\pi(\{x+1\}) = \pi(\{x\}) \frac{p(x, x+1)}{p(x+1, x)}.$$

- For random walk on a finite graph, if the chain chooses a neighboring vertex uniformly at random, then detailed balance suggests that vertices with larger degree should receive larger stationary weight.
- To show that a stationary distribution π is **attractive**, one shows that the distribution of X_n converges to π as $n \rightarrow \infty$, regardless of the initial state, meaning

$$\mathbb{P}(X_n = x) \xrightarrow{n \rightarrow \infty} \pi(\{x\}).$$

Problem 1. In a repeated Prisoner's Dilemma, player R plays randomly and player V is vengeful. At each time n , player R chooses either trust (T) or betray (B), with

$$\mathbb{P}(R_n = B) = \frac{1}{2}, \quad \mathbb{P}(R_n = T) = \frac{1}{2},$$

and the random choices R_n are independent over time. Player V starts by trusting. Once player R betrays at some time, player V betrays forever after that. More precisely, if $R_n = B$, then

$$V_m = B \quad \text{for all } m > n.$$

Find the stationary distribution of $\{(R_n, V_n)\}_{n \in \mathbb{N}}$ and that this stationary distribution is attractive.

Problem 2. A Markov chain $\{X_n\}_{n \geq 0}$ tracks the number of rabbits on an island, so the state space is $\{0, 1, 2, \dots\}$. Assume that for $x > 0$,

$$p(x, x+1) = \beta(x), \quad p(x, x-1) = \frac{1}{2}, \quad p(x, x) = \frac{1}{2} - \beta(x),$$

where $0 \leq \beta(x) \leq \frac{1}{2}$. At state 0, assume

$$p(0, 1) = \beta(0), \quad p(0, 0) = 1 - \beta(0),$$

where $0 \leq \beta(0) \leq 1$.

1. Find a stationary measure π satisfying detailed balance, expressed in terms of the function β .
2. Specialize your answer to the case $\beta(x) = \frac{1}{3}$ for every $x \geq 0$.
3. Now suppose instead that

$$\beta(0) = 0, \quad \beta(x) = \frac{1}{3} \text{ for } x \geq 1.$$

What happens to your stationary measure in this case?

Problem 3. Let $G = (V, E)$ be a finite connected graph. A Markov chain $\{X_n\}_{n \geq 0}$ moves on the vertex set V by choosing one of the neighboring vertices uniformly at random at each step. Thus, if $x \sim y$, then

$$p(x, y) = \frac{1}{\deg(x)},$$

and otherwise $p(x, y) = 0$.

1. Use detailed balance to show that if π is a stationary distribution, then

$$\frac{\pi(\{x\})}{\deg(x)}$$

is the same for every vertex $x \in V$.

2. Deduce that a stationary distribution is given by

$$\pi(\{x\}) = \frac{\deg(x)}{\sum_{v \in V} \deg(v)}.$$

3. Briefly explain why vertices of high degree are visited more often in the long run.