

MATH 171 Worksheet 3

Recurrence and transience

Review

Let $\{X_n\}_{n \geq 0}$ be a Markov chain with state space S .

- For $y \in S$, the first return time to y is

$$T_y := \min \{n \geq 1 : X_n = y\}.$$

If the chain starts from x , write

$$\rho_{xy} := \mathbb{P}_x(T_y < \infty).$$

- A state y is **recurrent** if $\rho_{yy} = 1$. It is **transient** if $\rho_{yy} < 1$.
- We say that x **communicates with** y , written $x \rightarrow y$, if $\rho_{xy} > 0$.
- A set $C \subseteq S$ is **closed** if for every $x \in C$ and $y \notin C$,

$$p(x, y) = 0.$$

It is **irreducible** if every two states in C communicate.

- Facts for finite-state chains:
 - If x is recurrent and $x \rightarrow y$, then y is recurrent.
 - Every finite closed irreducible set consists entirely of recurrent states.
 - The state space can be decomposed as

$$S = T \cup R_1 \cup \cdots \cup R_k,$$

where T is the set of transient states and the R_i are disjoint closed irreducible recurrent classes.

1 Problems

Problem 1.1. Consider the Markov chain on $S = \{0, 1, 2, 3, 4\}$ with transition probabilities

$$\begin{array}{ll} p(0, 0) = 1, & p(4, 4) = 1, \\ p(1, 0) = \frac{2}{3}, & p(1, 2) = \frac{1}{3}, \\ p(2, 1) = \frac{1}{2}, & p(2, 3) = \frac{1}{2}, \\ p(3, 2) = \frac{1}{4}, & p(3, 4) = \frac{3}{4}, \end{array}$$

and all other transition probabilities are 0.

1. Draw the transition graph.
2. Decide which states are recurrent and which are transient.
3. For each transient state $i \in \{1, 2, 3\}$, find an event of positive probability on which the chain never returns to i .

Problem 1.2. Consider the Markov chain on $S = \{1, 2, 3, 4, 5, 6, 7\}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

1. Find all communicating classes.
2. Classify every state as recurrent or transient.
3. Find the decomposition

$$S = T \cup R_1 \cup \dots \cup R_k.$$

Problem 1.3. Consider the Markov chain on $S = \{0, 1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

Let

$$T_0 := \min \{n \geq 1 : X_n = 0\}.$$

1. Show that from any state $x \in S$, the probability of hitting state 0 in the next step is at least $1/4$.
2. Show that

$$\mathbb{P}_0(T_0 > n) \leq \left(\frac{3}{4}\right)^n \quad \text{for all } n \geq 1.$$

3. Conclude that state 0 is recurrent.
4. Conclude that every state is recurrent.