

MATH 171 Worksheet 2

First examples of Markov chains

Review.

A discrete-time stochastic process is a sequence of random variables $\{X_n\}_{n \geq 0}$. A process with state space S is a time-homogeneous Markov chain if for all $n \geq 0$ and all states $i_0, \dots, i_n, j \in S$,

$$\mathbb{P}(X_{n+1} = j \mid X_n = i_n, \dots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i_n).$$

For a finite-state Markov chain, the one-step transition probabilities are

$$p_{ij} := \mathbb{P}(X_{n+1} = j \mid X_n = i),$$

and these form the transition matrix $P = (p_{ij})_{i,j \in S}$. Each row of P is a probability vector, so

$$p_{ij} \geq 0, \quad \sum_{j \in S} p_{ij} = 1.$$

If $\mu^{(n)}$ denotes the distribution of X_n written as a row vector, then

$$\mu^{(n+1)} = \mu^{(n)} P, \quad \mu^{(n)} = \mu^{(0)} P^n.$$

We say that μ is a *stationary distribution* of the Markov chain with transition matrix P iff $\mu P = \mu$, meaning μ is an eigenvector of P with eigenvalue one.

1 Problems

Problem 1.1. Let $W_n \in \{S, R\}$ denote the weather on day n , where S means sunny and R means rainy. Suppose the law of tomorrow's weather depends on whether the last two days had the same weather:

- if $W_n = W_{n-1}$, then $\mathbb{P}(W_{n+1} = W_n \mid W_n, W_{n-1}) = \frac{4}{5}$,
- if $W_n \neq W_{n-1}$, then $\mathbb{P}(W_{n+1} = W_n \mid W_n, W_{n-1}) = \frac{2}{5}$.

1. Explain why $\{W_n\}_{n \geq 0}$ is not a Markov chain on the state space $\{S, R\}$.
2. Define a new process X_n that records enough information to make the process Markov.
3. Write down the state space and transition matrix for $\{X_n\}_{n \geq 1}$.

Problem 1.2. A ball is hidden under one of three cups. The cups are shuffled repeatedly. After each move of the shuffler, you either still know where the ball is or you lose track of it. If you know where the ball is before a move, then after that move you still know where it is with probability 0.9 and you lose track of it with probability 0.1. Once you lose track, you never recover it. If you have lost track, your best strategy is to guess uniformly among the three cups.

Let $Y_n \in \{T, L\}$ denote your status after n moves, where T means you are still tracking the ball and L means you have lost track.

1. Write down the transition matrix for $\{Y_n\}_{n \geq 0}$.

2. Assume $Y_0 = T$. Find the probability that after n moves you guess the correct cup.
3. What is the smallest integer n for which this probability is less than $\frac{1}{2}$?
4. Prove that no matter the distribution of Y_0 we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n = T) = \frac{1}{3}.$$

Problem 1.3. At time 0 there is one bacteria in the Petri dish. Each bacteria independently produces either 0 children with probability $\frac{1}{3}$ or 2 children with probability $\frac{2}{3}$. Let Z_n be the number of bacteria in generation n .

Show that

$$\mathbb{E}[Z_n] = \left(\frac{4}{3}\right)^n \quad \text{for all } n \geq 0.$$

Hint: Prove the formula inductively, starting by disintegration of measure of Z_{n+1} with respect to Z_n :

$$\mathbb{E}[Z_{n+1}] = \sum_{k \in \mathbb{N}} \mathbb{P}(Z_n = k) \mathbb{E}[Z_{n+1} \mid Z_n = k].$$