

MATH 171 Worksheet 10

Continuous-time Markov chains

Review. A continuous-time Markov chain has transition rates $q(i, j)$ for $i \neq j$. If the chain is in state i , its total rate of leaving is

$$q_i = \sum_{j \neq i} q(i, j).$$

The holding time in state i is exponential with rate q_i . When the chain leaves i , it jumps to j with probability

$$\frac{q(i, j)}{q_i}.$$

The generator is the matrix Q defined by

$$Q(i, j) = q(i, j) \quad \text{for } i \neq j, \quad Q(i, i) = -\sum_{j \neq i} q(i, j).$$

Each row of Q sums to 0.

A probability vector π is stationary if

$$\pi Q = 0, \quad \sum_i \pi(i) = 1.$$

A sufficient condition for stationarity is detailed balance:

$$\pi(i)q(i, j) = \pi(j)q(j, i) \quad \text{for all } i, j.$$

For a target set A , let $T_A = \inf \{t \geq 0 : X_t \in A\}$. If

$$g(i) = \mathbb{E}_i [T_A],$$

then

$$g(i) = 0 \quad \text{for } i \in A,$$

and for $i \notin A$,

$$\sum_j Q(i, j)g(j) = -1.$$

Problem 1. A web server is modeled by the states

$$0 = \text{idle}, \quad 1 = \text{busy}, \quad 2 = \text{down}.$$

Jobs arrive to an idle server at rate 3. A busy server completes its job at rate 4 and breaks down at rate 1. A down server is repaired at rate 2 and returns idle.

Write the generator Q . If the server is busy, find the expected holding time in that state and the probabilities of the two possible next states.

Problem 2. A small call center has room for at most three active calls. Calls arrive at rate 2 whenever fewer than three calls are active. Each active call ends independently at rate 1. Let X_t be the number of active calls.

Use detailed balance to find the stationary distribution of X_t . What is the long-run fraction of time that all three slots are full?

Problem 3. A machine has three states:

$$0 = \text{working}, \quad 1 = \text{degraded}, \quad 2 = \text{failed}.$$

The transition rates are

$$0 \rightarrow 1 \text{ at rate } 1, \quad 1 \rightarrow 0 \text{ at rate } 2, \quad 1 \rightarrow 2 \text{ at rate } 3.$$

The failed state is absorbing.

Find the expected time to failure starting from the working state.