

MATH 171 — Worksheet 1

Probability review

Review

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, meaning a triplet of a set Ω , the σ -algebra $\mathcal{F}$ of measurable sets, called *events* and the measure $\mathbb{P}$ on Ω , which is formally a function $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$.

Events and basic identities. If events $A, B \in \mathcal{F}$, then A^c denotes the complement of A , and

$$\begin{aligned}\mathbb{P}(\emptyset) &= 0, \\ \mathbb{P}(A^c) &= 1 - \mathbb{P}(A), \\ \mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).\end{aligned}$$

If events $A_1, \dots, A_n$ are pairwise disjoint, then

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i).$$

For any countable family of events $\{A_n\}_{n \in \mathbb{N}}$ we have

$$\begin{aligned}\mathbb{P}\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \leq n} A_k\right), \\ \mathbb{P}\left(\bigcap_{n \in \mathbb{N}} A_n\right) &= \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{k \leq n} A_k\right).\end{aligned}$$

This property is called *continuity of measure* and holds for non-probability measures as well. Note that the order in which events are put is irrelevant.

Conditional probability. If $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Equivalently,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A | B) \mathbb{P}(B) = \mathbb{P}(B | A) \mathbb{P}(A).$$

Independence. Events A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B).$$

Random variables $X : \Omega \rightarrow \mathcal{X}$ and $Y : \Omega \rightarrow \mathcal{Y}$ are independent if for all measurable sets $S_X \subseteq \mathcal{X}$, $S_Y \subseteq \mathcal{Y}$,

$$\mathbb{P}(X \in S_X, Y \in S_Y) = \mathbb{P}(X \in S_X) \mathbb{P}(Y \in S_Y).$$

Law of total probability and Bayes' rule. If $B_1, \dots, B_n$ form a partition of Ω and $\mathbb{P}(B_i) > 0$ for all i , then

$$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A | B_i) \mathbb{P}(B_i).$$

If also $\mathbb{P}(A) > 0$, then for each j ,

$$\mathbb{P}(B_j | A) = \frac{\mathbb{P}(A | B_j) \mathbb{P}(B_j)}{\sum_{i=1}^n \mathbb{P}(A | B_i) \mathbb{P}(B_i)}.$$

Random variables, distributions, expectation. A random variable is a function $X : \Omega \rightarrow \mathbb{R}$. Its distribution function is

$$F_X(x) = \mathbb{P}(X \leq x).$$

If X is discrete with pmf $p_X(x) = \mathbb{P}(X = x)$, then

$$\begin{aligned}\mathbb{E}[X] &= \sum_x x p_X(x), \\ \text{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.\end{aligned}$$

If X is continuous with density f_X , then

$$\begin{aligned}\mathbb{E}[X] &= \int_{\mathbb{R}} x f_X(x) dx, \\ \text{Var}(X) &= \int_{\mathbb{R}} (x - \mathbb{E}[X])^2 f_X(x) dx.\end{aligned}$$

For a function g ,

$$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$$

for discrete X , and

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f_X(x) dx$$

for continuous X , whenever the expression is well-defined.

Joint and conditional distributions. If (X, Y) is discrete, the joint pmf is

$$p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y).$$

If (X, Y) is continuous with joint density $f_{X,Y}$, then the marginals are

$$\begin{aligned}f_X(x) &= \int_{\mathbb{R}} f_{X,Y}(x, y) dy, \\ f_Y(y) &= \int_{\mathbb{R}} f_{X,Y}(x, y) dx,\end{aligned}$$

and, whenever $f_X(x) > 0$,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

For continuous random variables, independence is equivalent to

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

on the support.

Problems

1. A box is chosen at random. With probability $1/3$ choose box B_1 , which contains 3 red balls and 1 blue ball. With probability $2/3$ choose box B_2 , which contains 1 red ball and 3 blue balls. From the chosen box, draw two balls without replacement.

Let A be the event that the first draw is red, and let C be the event that the second draw is red.

- (a) Compute $\mathbb{P}(A)$, $\mathbb{P}(C)$, and $\mathbb{P}(A \cap C)$.
- (b) Are A and C independent?
- (c) Compute $\mathbb{P}(B_1 | C)$.

2. A fair coin is tossed three times. Let X be the total number of heads, and let $Y = \mathbf{1}_{\{\text{first toss is H}\}}$.

- (a) Write down the joint pmf of (X, Y) .
- (b) Find the marginal distribution of X .

- (c) Compute $\mathbb{E}[X]$, $\text{Var}(X)$, and $\mathbb{E}[X \mid Y = 1]$.
- (d) Are X and Y independent?

3. Let (X, Y) have joint density

$$f_{X,Y}(x, y) = 2$$

for $0 < y < x < 1$, and $f_{X,Y}(x, y) = 0$ otherwise.

- (a) Find the marginal densities f_X and f_Y .
- (b) Find the conditional density $f_{Y|X=x}(y)$.
- (c) Compute $\mathbb{P}(Y < X/2)$.
- (d) Are X and Y independent?