# MAXIMAL INDEXES OF FLAG VARIETIES FOR SPIN GROUPS

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Abstract. We establish the sharp upper bounds on the indexes for most of the twisted flag varieties under the spin groups $\mathrm{Spin}(n)$.

Contents



# Part 1. Odd dimension

## 1. Introduction

Let $F$ be any field and let $V$ be an $F$-vector space of finite odd dimension $2n+1$ for some integer $n > 0$. Let $q\colon V \to F$ be a *non-degenerate* quadratic form (see [6, Definition 7.17]). For any $m = 1, \dots, n$, the $m$th orthogonal grassmannian $X_m$ of $q$ is defined as the variety of $m$-dimensional totally isotropic subspaces in $V$. Thus, $X_m$ is a closed subvariety inside the usual $m$-grassmannian of the vector space $V$. The two extremes here are studied the most in the literature: the projective quadric $X_1$ and the highest orthogonal grassmannian

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$X_n$ (see, e.g., [15] and [16] by A. Vishik; see [6] for a more exhaustive list of results and references).

For quadratic forms of even dimension, the similar varieties and the question formulated below will be considered in the second part of the paper.

The variety $X_m$ has a rational point if and only if the Witt index $i_W(q)$ of the quadratic form $q$ is at least $m$. The index $i(X_m)$ of the variety $X_m$ (or of any other algebraic variety) is defined as the g.c.d. of the degrees of its closed points. Since $q$ can be completely split by a multiquadratic field extension, $i(X_m)$ is a power of 2. Namely, $i(X_m)$ is the maximal 2-power dividing the degree of every finite field extension $L/F$ satisfying $i_W(q_L) \geq m$.

The question considered in this paper is as follows: given $n$ and $m$, what is the maximal value of $i(X_m)$ when $F$, $V$, $q$ vary (more precisely, we let $F$ vary over all field extensions of a fixed field) and the Clifford invariant of $q$ (i.e., the Brauer class $[C_0(q)]$ of the even Clifford algebra $C_0(q)$) is trivial? Since this maximal value is known to be realized at any generic quadratic forms with trivial Clifford invariant (defined below), what we want is just to determine $i(X_m)$ in the case of a generic $q$. Note that if we drop the condition on triviality of the Clifford invariant, the answer to the modified question becomes $2^m$ for any $m$. Clearly, this is an upper bound for the original question.

Multiplication of $q$ by a non-zero element of $F$ does not change the varieties $X_m$. In the case of trivial Clifford invariant, the similarity class of the quadratic form $q$ is given by a $\mathrm{Spin}(2n+1)$-torsor $E$ over $F$. Instead of the similarity class, one sometimes prefers to speak about its (unique up to isomorphism) discriminant 1 representative (see [6] for definition of discriminant in arbitrary characteristic). (And any $\mathrm{Spin}(2n+1)$-torsor over $F$ yields a similarity class of a $(2n+1)$-dimensional quadratic form with Brauer trivial even Clifford algebra. By the way, saying "generic $q$" above we meant that it was given by a *generic torsor* defined as the generic fiber of the quotient morphism $\mathrm{GL}(N) \to \mathrm{GL}(N)/\mathrm{Spin}(2n+1)$ for an embedding $\mathrm{Spin}(2n+1) \hookrightarrow \mathrm{GL}(N)$ with some $N \geq 1$.)

Moreover, one has $X_m \simeq E/P$ for an appropriate parabolic subgroup $P \subset \mathrm{Spin}(2n+1)$. In other terms, $X_m$ is the flag variety $\mathrm{Spin}(2n+1)/P$ twisted by $E$. For an arbitrary proper parabolic subgroup $P \subset \mathrm{Spin}(2n+1)$, the twisted flag variety $E/P$ is the variety of flags of totally isotropic subspaces in $V$ of some dimensions $1 \leq m_1 < \cdots < m_r \leq n$ (with some $r \geq 1$). The index of this variety coincides with $i(X_{m_r})$ and therefore does not require additional investigation.

The case of the highest orthogonal grassmannian $X_n$ has been done by B. Totaro in [14]. (Note that $i(X_n) = i(X_{n-1}) = i(X_{n-2})$ if defined.) The answer and the proof there are quite complicated. For generic $q$, the integer $i(X_n)$ is equal to $2^t$, where $t$ is approximately $n - 2\log_2(n)$.

For the other extreme – the projective quadric $X_1$, the answer is much simpler: it is just $2 = 2^m$ if we only look at the forms $q$ of dimension at least 7. In other terms, the evident upper bound $i(X_m) \leq 2^m$ turns out to be sharp here. Note that this answer is equivalent to the following assertion: there are anisotropic forms $q$ with trivial Clifford invariant in every dimension $\dim q \geq 7$. (We do not know a simple proof for this statement. Several different proofs are presented in the appendix.)

In this paper we show for arbitrary $m \le n$ that the evident upper bound $2^m$ is sharp provided that $2^{n-m} > \dim X_m$ (see Theorem 4.2). In fact, the number $m$ only needs to be slightly smaller than $t$. For instance, for $n = 2021$ the equality $i(X_m) = 2^m$ holds for generic $q$ if $m \le 2000$ and $t = 2001$ by [14, Theorem 0.1]. Since $i(X_m) \le i(X_n)$ for any $m$, the formula $\max i(X_m) = 2^m$ fails for $m > 2001$. So, $m = 2001$ is the only value of $m$ for which Theorem 4.2 and [14, Theorem 0.1] together do not determine if it satisfies the formula.

Actually, an overwhelming majority of the natural numbers $n$ have the same property: the formula $i(X_m) = 2^m$ holds for $m < t$ and fails for $m > t$. More precisely, the proportion of such $n < N$ tends to 1 when $N \to \infty$. Note that for such numbers $n$, we know that $m \mapsto i(X_m)$ is an increasing function with $i(X_{t-1}) = 2^{t-1}$ and $i(X_n) = 2^t$, but we don't know where exactly this function jumps from $2^{t-1}$ to $2^t$.

A conjecture in the direction of Theorem 4.2 has been suggested for quadratic forms of even dimension in [8] as a possible enhancement of some results from [2]. Extended to quadratic forms of odd dimension it would affirm that (outside of small $n$) the maximal value of $i(X_m)$ is $2^m$ in the case of $m \le n/2$. Our main result here not only confirms this for $n \ge 13$ but also drastically extends the range of $m$ for larger $n$.

In the next section we suggest a method of bounding the indexes of twisted flag varieties under any split reductive algebraic group. On the example of the spin groups, we see then that this method is capable to provide interesting results.

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## 2. General considerations

Let $G$ be a split reductive group over a field $F$. Let $T \subset B \subset P \subset G$ be a split maximal torus, a Borel subgroup, and a parabolic subgroup of $G$. There is a canonical homomorphism of graded rings

$$f_P\colon \operatorname{CH}(BP) \to \operatorname{CH}(G/P).$$

The ring on the left is the Chow ring of the classifying space of $P$, originally defined in [13], which coincides with the $P$-equivariant Chow ring $\operatorname{CH}_P(\operatorname{Spec} F)$ of the point (see [5]). And the homomorphism $f_P$ is simply the pull-back $\operatorname{CH}_P(\operatorname{Spec} F) \to \operatorname{CH}_P(G) = \operatorname{CH}(G/P)$ with respect to the structure morphism of the $F$-variety $G$.

For $d := \dim G/P$, we have $\operatorname{CH}^d(G/P) = \mathbb{Z}$ and the cokernel of $\operatorname{CH}^d(BP) \to \operatorname{CH}^d(G/P)$ has a finite order $i_P \ge 1$. It is shown in [11, Theorem 6.4] that the integer $i_P$ thus defined is the index of the variety $E/P$, where $E$ is any generic $G$-torsor (over a field extension of $F$). So, for $G$ a spin group, this will be the integer of our interest.

In fact, [11, Theorem 6.4] tells that the image of the homomorphism $\operatorname{CH}(BP) \to \operatorname{CH}(G/P)$ of the total Chow rings coincides with the image of $\operatorname{CH}(E/P) \to \operatorname{CH}(G/P)$ for generic $E$ and is contained in the latter image for arbitrary (not necessarily generic) $E$.

If the parabolic subgroup $P$ is special, the ring $\operatorname{CH}(BP)$ is identified with $\operatorname{CH}(BT)^W = \mathsf{S}(\hat{T})^W$, where $W$ is the Weyl group of $P$, $\hat{T}$ is the group of characters of $T$, and $\mathsf{S}$ is the symmetric ring functor (see [5, Proposition 6] together with [10, Proof of Proposition 6.1]). This makes it possible to compute the index $i_P$ (and constitutes the starting point in obtaining the result of [14]).

For instance, $B$ is a special parabolic subgroup with trivial Weyl group so that we have

$$f_B\colon S(\hat{T}) \to \mathrm{CH}(G/B).$$

Unfortunately, for arbitrary (not necessarily special) $P$, a computation of $\mathrm{CH}(BP)$ is sometimes (or rather most of the time) out of reach. However there still is a canonical homomorphism of graded rings $\mathrm{CH}(BP) \to S(\hat{T})^W$. (It is neither injective nor surjective in general, but becomes an isomorphism after tensoring with $\mathbb{Q}$.) Indeed, the restriction of action yields a homomorphism $\mathrm{CH}(BP) \to \mathrm{CH}(BT) = S(\hat{T})$ whose image consists of $W$-invariant elements.

**Lemma 2.1.** *There is one and unique homomorphism of graded rings*

$$f'_P\colon S(\hat{T})^W \to \mathrm{CH}(G/P)$$

*such that the composition*

$$\mathrm{CH}(BP) \longrightarrow S(\hat{T})^W \xrightarrow{f'_P} \mathrm{CH}(G/P)$$

*is $f_P$. The square*

$$\begin{array}{ccc} S(\hat{T}) & \xrightarrow{f_B = f'_B} & \mathrm{CH}(G/B) \\ \uparrow & & \uparrow \\ S(\hat{T})^W & \xrightarrow{f'_P} & \mathrm{CH}(G/P) \end{array}$$

*commutes.*

*Proof.* The group $\mathrm{CH}(G/P)$ is (torsion) free. The cokernel (as well as the kernel) of $\mathrm{CH}(BP) \to S(\hat{T})^W$ is torsion, [5, Proposition 6] (reduction to the Levi subgroup of $P$ is explained in [10, Proof of Proposition 6.1]). This implies unicity of $f'_P$.

By [1, Proposition 20.5], for any extension field $K/F$, the map $G(K) \to (G/P)(K)$ of the sets of $K$-points is surjective. Applying this property to the function field of the variety $G/P$, one sees that the $P$-torsor given by the generic fiber of the quotient map $G \to G/P$ is trivial. In particular, the generic fiber of the projection $\pi\colon G/B \to G/P$ has a rational point. The class $x \in \mathrm{CH}(G/B)$ of its closure in $G/B$ satisfies $\pi_*(x) = 1$. By projection formula, for any $y \in \mathrm{CH}(G/P)$ we have $\pi_*(\pi^*(y) \cdot x) = y \cdot \pi_*(x) = y$. It follows that

$$\pi^*\colon\ \mathrm{CH}(G/P) \to \mathrm{CH}(G/B)$$

is a split monomorphism. In particular, its cokernel is (torsion) free (see also [7, Remark 3.3]).

Given any

$$x \in S(\hat{T})^W \subset S(\hat{T}),$$

we can find a nonzero integer $n$ with $nx$ in the image of $\mathrm{CH}(BP)$. The element $f_B(x) \in \mathrm{CH}(G/B)$ has then the property $nf_B(x) = f_B(nx) \in \mathrm{CH}(G/P)$. Consequently, $f_B(x) \in \mathrm{CH}(G/P) \subset \mathrm{CH}(G/B)$. So, the restriction of the map $f_B$ to $S(\hat{T})^W$ yields the required map $f'_P$. □

**Remark 2.2.** By [13, Theorem 1.3], the ring $\mathrm{CH}(BP)$ can be viewed as the ring of all assignments to every $P$-torsor over a smooth variety $X$ of an element in $\mathrm{CH}(X)$ (natural in $X$). The ring $S(\hat{T})^W$ has a similar interpretation with "$P$-torsor" replaced by "Zariski locally trivial $P$-torsor" ([4, Theorem 1]). In this interpretations, the homomorphism $\mathrm{CH}(BP) \to S(\hat{T})^W$ is given by restriction of assignments. The homomorphisms $f_P$ and $f'_P$ are given by evaluation of assignments at the $P$-torsor $G \to G/P$; here $f'_P$ is well defined because this $P$-torsor is Zariski locally trivial.

It follows that the order of $\mathrm{Coker}\big(S^d(\hat{T})^W \to \mathrm{CH}^d(G/P)\big)$ is a lower bound on $i_P$. For $G$ a spin group, this lower bound is going to satisfy our needs.

## 3. Computation of invariants

Let $R = \mathbb{Z}[x_1, \dots, x_m, y_1, \dots, y_l]$ be the polynomial ring over $\mathbb{Z}$ in variables $x_1, \dots, x_m$ and $y_1, \dots, y_l$ with some $m, l \geq 1$. We consider the $R$-algebra $R[z]$ with a generator $z$ subject to the relation

$$2z = x_1 + \cdots + x_m + y_1 + \cdots + y_l.$$

Let $A := (\mathbb{Z}/2\mathbb{Z})^{\times l}$ be the direct product of $l$ copies of the group $\mathbb{Z}/2\mathbb{Z}$ acting on $R$ as follows: for any $i = 1, \dots, l$, the $i$th copy of $\mathbb{Z}/2\mathbb{Z}$ acts by changing the sign of $y_i$ and trivially on the remaining variables. Note that $R^A$ is the subring in $R$ generated by $x_1, \dots, x_m$ and the squares $y_1^2, \dots, y_l^2$.

The action of $A$ on $R$ extends uniquely to $R[z]$: the 1 of the $i$th copy of $\mathbb{Z}/2\mathbb{Z}$ maps $z$ to $z - y_i$.

The orbit of the element $z$ under this action consists of $2^l$ elements $z - \sum_{i \in I} y_i$, where $I$ runs over all subsets of $\{1, \dots, l\}$. We write $\tilde{z} \in R[z]^A$ for the product of the elements in the orbit of $z$.

We construct $l$ more $A$-invariant elements $f_0, \dots, f_{l-1}$ in $R[z]$. We set

$$f_0 := 2z - y_1 - \cdots - y_l = x_1 + \cdots + x_m \in R^A.$$

Assume that for some $k = 0, \dots, l-2$ the element $f_k$ is already constructed and has the shape

$$f_k = 2z \cdot g_k + a_1 + \cdots + a_s, \tag{3.1}$$

where $g_k$ is a polynomial with integer coefficients in $z, y_1, \dots y_l$ and where $a_1, \dots, a_s$ for some $s \geq 0$ are monomials in $y_1, \dots, y_l$. Then we define $f_{k+1}$ as one half of the difference

$$f_k^2 - (a_1^2 + \cdots + a_s^2) = 2\Big(2z\big(zg_k^2 + (a_1 + \cdots + a_s)g_k\big) + \sum_{i \neq j} a_i a_j\Big).$$

Note that the new polynomial $f_{k+1}$ has the shape (3.1) allowing to continue the procedure.

We will also consider the induced action of $A$ on the quotient ring $R[z]/2R[z]$. This quotient is the polynomial ring $S[z]$ in the variable $z$ (which is the class of the above $z$ but is subject to no relations anymore) over the ring

$$S = \mathbb{F}_2[x_1, \dots, x_m, y_1, \dots, y_l]/(x_1 + \cdots + x_m + y_1 + \cdots + y_l),$$

which is itself a polynomial ring (over the field $\mathbb{F}_2$ in $m + l - 1$ variables). Note that the action of $A$ on $S$ is trivial and that the element $\tilde{z} \in S[z]$ (the class modulo 2 of the above $\tilde{z} \in R[z]$) is (still) $A$-invariant. In other terms, $S[z]^A \supset S[\tilde{z}]$.

**Lemma 3.2.** $S[z]^A = S[\tilde{z}]$.

*Proof.* Given any $A$-invariant $f \in S[z]$, viewed as a polynomial in $z$, we remove its constant term. Then $f$ is divisible by $z$ and, therefore, by every factor in the definition of $\tilde{z}$. Since all these factors are distinct primes of the factorial ring $S[z]$, $f$ is divisible by $\tilde{z}$. Since $S$ is an integral domain, the quotient $f/\tilde{z}$ is $A$-invariant as well and so – by induction on degree – is a polynomial in $\tilde{z}$. Therefore $f \in S[\tilde{z}]$. □

**Proposition 3.3.** *The $R^A$-algebra $R[z]^A$ is generated by the $l$ elements $\tilde{z}, f_1, \dots, f_{l-1}$.*

*Proof.* It suffices to prove the result for $m = 1$. Indeed, we can view $R[z]$ as the polynomial ring over $\mathbb{Z}[z, y_1, \dots, y_l]$ in the variables $x_2, \dots, x_m$. A polynomial here is $A$-invariant if and only if all its coefficients are. Moreover, our $l$ potential generators are in the coefficient ring $\mathbb{Z}[z, y_1, \dots, y_l]$.

So, below we work with the case $m = 1$. From now on, we view the ring $R[z]$ as the ring of polynomials over $R' := \mathbb{Z}[y_1, \dots, y_l]$ in the (independent!) variable $z$.

Let $f \in R'[z]$ be $A$-invariant. We prove that $f$ is in the $R^A$-subalgebra generated by $\tilde{z}, f_1, \dots, f_{l-1}$ using induction on $\deg f$.

If $\deg f \leq 0$, then $f \in R'^A \subset R^A$. Below we assume that $\deg f > 0$.

If $\deg f \geq 2^l$, then we let $h$ be the highest power of $\tilde{z}$ with $\deg h \leq \deg f$ and we divide $f$ by $h$ with remainder. The division goes through because the leading coefficient of $h$ is 1. Since $f$ and $h$ are $A$-invariant, so are the partial quotient and the remainder. Besides, their degrees are smaller than $\deg f$.

We are left with the case $0 < \deg f < 2^l$. By Lemma 3.2, all coefficients of $f$ besides the constant term are even (i.e., divisible by 2). We divide $f$ with remainder by $f_k$ with the highest $k \in \{0, 1, \dots, l-1\}$ such that $2^k = \deg f_k \leq \deg f$. By formula (3.1) as well as by Lemma 3.2, all coefficients of $f_k$ besides the constant term are also even. Moreover, the leading coefficient of $f_k$ is 2. Since the degree of $f_k$ is higher than half of the degree of $f$, the division with remainder goes through. The partial quotient and the remainder are $A$-invariant and have degrees smaller than $\deg f$. □

For $n := m + l$, let $c_1, \dots, c_n \in R$ be the elementary symmetric polynomials in $x_1, \dots, x_m, y_1, \dots, y_l$. For convenience, we additionally set $c_i := 0$ for $i > n$. The ring $C$ of polynomials in $c_1, \dots, c_n$ over $\mathbb{Z}$ is a subring in $R$. Let $J$ be the ideal of $C$ generated by

$$2\Big(c_{2i} - c_1 c_{2i-1} + \dots + (-1)^{i-1} c_{i-1} c_{i+1}\Big) + (-1)^i c_i^2 \tag{3.4}$$

with $i = 1, \dots, n$. Let $I$ be the ideal of the ring $R[z]$ consisting of the elements such that after multiplication by an appropriate nonzero integer they are in the ideal of $R[z]$ generated by $J$.

**Lemma 3.5.** *For every $k = 0, \dots, l-1$, the element $f_k \in R[z]$ is congruent modulo $I$ to an element of $R$.*

*Proof.* Inducting on $k$, we will prove the following (stronger) statement: the element $2zg_k$ is congruent modulo $I$ to an element of $C'[y_1, \dots, y_l] \subset R$, where $C'$ is the ideal of the polynomial ring $C = \mathbb{Z}[c_1, \dots, c_n]$ consisting of the polynomials without constant term.

The element $2zg_0 = 2z = c_1$ is in $C'[y_1,\dots,y_l]$. Assume that for some $k = 0,\dots,n-2$, the element $2zg_k$ is congruent modulo $I$ to an element of $C'[y_1,\dots,y_l]$. Then $(2zg_k)^2$ is congruent modulo $I$ to a square of an element of $C'[y_1,\dots,y_l]$. Note that the square of any element of $C'$ is congruent modulo $J$ to an element of $2C'$ (see formula (3.4) and [14, discussion after Lemma 4.1]). Therefore $2z^2g_k^2 = (2zg_k)^2/2$ is congruent modulo $I$ to an element of $C'[y_1,\dots,y_l]$ and it follows that the element

$$2zg_{k+1} = 2z\big(zg_k^2 + (a_1+\dots+a_s)g_k\big)$$

satisfies the same property. □

## 4. Main result

In this section, $G$ is the split spin group $\mathrm{Spin}(2n+1)$ for some $n \geq 1$ over an arbitrary field $F$. As in [11, §8.2], we construct $G$ out of a split quadratic form $q$ defined on a vector space $V$ with a basis given by a vector $g$ and vectors $e_i, f_i$, $i = 1,\dots,n$, where $e_i, f_i$ are pairwise orthogonal hyperbolic pairs orthogonal to $g$. Let us fix some $m \in \{1,\dots,n\}$, consider the $m$-dimensional totally isotropic subspace generated by $e_1,\dots,e_m$ and let $P \subset G$ be its stabilizer. Then $P$ is a parabolic subgroup in $G$ and the variety $G/P$ is the $m$th orthogonal grassmannian $X_m$ of $q$.

We take for $T \subset P$ the split maximal torus, mapped under the isogeny $G \to G/\mu_2 = \mathrm{O}^+(2n+1)$ (with the special orthogonal group) to the split maximal torus $T' := \mathbb{G}_{\mathrm{m}}^n \hookrightarrow \mathrm{O}^+(2n+1)$ given by $t(e_i) = t_ie_i$, $t(f_i) = t_i^{-1}f_i$, and $t(g) = g$, where $t = (t_1,\dots,t_n) \in \mathbb{G}_{\mathrm{m}}^n(F)$. We have an exact sequence

$$1 \to \mu_2 \to T \to T' \to 1.$$

Writing $x_1,\dots,x_n$ for the standard basis of $\mathbb{Z}^n = \hat{T}'$, we therefore have $\hat{T} = \hat{T}' + \mathbb{Z}z$, where $z := (x_1+\dots+x_n)/2$.

Let us set $l := n-m$. The Weyl group $W$ of $P$ is the direct product of the symmetric group $S_m$ and the Weyl group of $\mathrm{O}^+(2l+1)$, the latter being a semidirect product of $S_l$ by $(\mathbb{Z}/2\mathbb{Z})^l$. The action of $W$ on $\hat{T}'$ is given by the action of $S_m$ by permutations of $x_1,\dots,x_m$, the action of $S_l$ by permutation of $x_{m+1},\dots,x_n$, and the action of the $i$th copy of $\mathbb{Z}/2\mathbb{Z}$ for $i = 1,\dots,l$ by changing the sign of $x_{m+i}$. This action extends uniquely to an action of $W$ on $\hat{T}$.

To comply with requirements of the previous section, we assume that $m < n$. Recall that we wrote $A$ for $(\mathbb{Z}/2\mathbb{Z})^{\times l}$ in the previous section. Let us identify $\mathsf{S}(\hat{T}')$ with the ring $R$ from there by identifying $x_{m+i}$ with $y_i$ for $i = 1,\dots,l$. The action of $A$ we had there is the restriction of the action of $W \supset A$ we have now. The element $z$ from the previous section corresponds to the element $z$ introduced here and $\mathsf{S}(\hat{T}) = R[z]$. The product $\tilde{z}$ from the previous section becomes an element of $\mathsf{S}^{2^l}(\hat{T})$.

The parabolic subgroup $P$ contains the Borel subgroup $B$ of $G$ defined as the stabilizer of the flag of the totally isotropic subspaces

$$\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \dots \subset \langle e_1,\dots,e_n\rangle\,.$$

We are going to study the image of the composition $\mathsf{S}(\hat{T})^W \hookrightarrow \mathsf{S}(\hat{T}) \to \mathrm{CH}(G/B)$. Note that the variety $G/B$ is the variety of complete flags of totally isotropic subspaces in $V$ and the image of $R = \mathsf{S}(\hat{T}') \hookrightarrow \mathsf{S}(\hat{T}) \to \mathrm{CH}(G/B)$ is the subring $C \subset \mathrm{CH}(G/B)$

generated by the first Chern classes of the tautological line bundles on $G/B$ (c.f. [9]). Note that these Chern classes are the images of $x_1, \dots, x_n$ (and will be denoted $x_1, \dots, x_n$ at a later point).

**Proposition 4.1.** *The image of $\mathsf{S}(\hat{T})^W$ in $\mathrm{CH}(G/B)$ is contained in the $C$-subalgebra generated by the image of $\tilde{z} \in \mathsf{S}^{2^l}(\hat{T})$.*

*Proof.* We note that $\mathsf{S}(\hat{T})^W \subset \mathsf{S}(\hat{T})^A$ and apply Proposition 3.3. The Chern classes of the tautological rank $n$ vector bundle on $G/B$ satisfy relations (3.4) (see [6, Formula 86.15]). Besides, $\mathrm{CH}(G/B)$ if (torsion) free. Consequently, the ideal $I \subset R[z] = \mathsf{S}(\hat{T})$ from Lemma 3.5 vanishes in $\mathrm{CH}(G/B)$ and we are done by Lemma 3.5. □

Let us recall the dimension formula for the variety $X_m$:

$$\dim X_m = m(m-1)/2 + m(2n+1-2m).$$

**Theorem 4.2.** *For $1 \le m \le n$, let $q$ be a generic $(2n+1)$-dimensional quadratic form with trivial Clifford invariant and let $X_m$ be its $m$th orthogonal grassmannian. Then $i(X_m) = 2^m$ provided that $2^{n-m} > \dim X_m$.*

*Proof.* Let $X$ be the variety of complete flags of totally isotropic subspaces for the form $q$. We are going to work with the varieties $X$, $X_1$, $X_m$, $X_n$ and with their base change $\bar{X}$, $\bar{X}_1$, $\bar{X}_m$, $\bar{X}_n$ to the algebraic closure of the base field.

As shown in [17, Statement 2.15], the class $[\mathsf{pt}] \in \mathrm{CH}(\bar{X}_m)$ of a rational point $\mathsf{pt}$ on $\bar{X}_m$ is equal to

$$[\mathsf{pt}] = \xi_m(l_0)\xi_m(l_1)\dots\xi_m(l_{m-1}),$$

where $l_i \in \mathrm{CH}_i(\bar{X}_1)$ is the class of a projective linear $i$-dimensional subspace on $\bar{X}_1$ and $\xi_m$ is the composition

$$\xi_m\colon\ \mathrm{CH}(\bar{X}_1) \to \mathrm{CH}(\bar{X}_{1,m}) \to \mathrm{CH}(\bar{X}_m)$$

of the pull-back followed by the push-forward with respect to the projections of the flag variety $X_{1,m} \subset X_1 \times X_m$. The image of $[\mathsf{pt}]$ under the pull-back $\mathrm{CH}(\bar{X}_m) \to \mathrm{CH}(\bar{X})$ with respect to the projection $X \to X_m$ can be computed via

**Lemma 4.3** ([17, Lemma 2.6])**.** *For any $i = 0, \dots, n-1$, the image of $\xi_m(l_i)$ under the pull-back $\mathrm{CH}(\bar{X}_m) \to \mathrm{CH}(\bar{X}_{m,m+1})$ is equal to $\xi_{m+1}(l_{i-1}) + \xi_{m+1}(l_i)x_{m+1}$. Here we view the ring $\mathrm{CH}(\bar{X}_{m,m+1})$ as a $\mathrm{CH}(\bar{X}_{m+1})$-algebra via the pull-back with respect to the projective bundle $X_{m,m+1} \to X_{m+1}$, $x_{m+1}$ is the first Chern class of the tautological line bundle on $X_{m,m+1}$, and $l_{-1} := 0$.*

It follows that the image of $[\mathsf{pt}]$ under $\mathrm{CH}(\bar{X}_m) \to \mathrm{CH}(\bar{X}_{m,m+1})$ equals

$$\Big(\xi_{m+1}(l_0)x_{m+1}\Big) \cdot \Big(\xi_{m+1}(l_0) + \xi_{m+1}(l_1)x_{m+1}\Big) \cdot \dots \cdot \Big(\xi_{m+1}(l_{m-2}) + \xi_{m+1}(l_{m-1})x_{m+1}\Big)$$

so that the coefficient at $x_{m+1}^m$ is $\xi_{m+1}(l_0)\xi_{m+1}(l_1)\dots\xi_{m+1}(l_{m-1})$. By iterating, the image of $[\mathsf{pt}]$ under the pull-back $\mathrm{CH}(\bar{X}_m) \to \mathrm{CH}(\bar{X})$ will be a polynomial in $x_{m+1}, \dots, x_n$ with coefficients in $\mathrm{CH}(\bar{X}_n)$ such that the coefficient at $x_{m+1}^m x_{m+2}^{m+1} \dots x_n^{n-1}$ is

$$\xi_n(l_0)\xi_n(l_1)\dots\xi_n(l_{m-1}) \in \mathrm{CH}(\bar{X}_n).$$

By Proposition 4.1, since $\tilde{z} \in \mathcal{S}^{2^{n-m}}(\hat{T})$ and $2^{n-m} > \dim X_m$, every element $a$ in the image of the composition

$$\mathrm{CH}(X_m) \to \mathrm{CH}(\bar{X}_m) \to \mathrm{CH}(\bar{X})$$

is a polynomial (with integer coefficients) in the first Chern classes $x_1, \ldots, x_n$ of the tautological line bundles on $X$. By Lemma 4.4, the element $a$ can be written uniquely as a polynomial in $x_1, \ldots, x_n$ with coefficients in $\mathrm{CH}(\bar{X}_n)$, where each $x_i$ appears only in degrees $< i$. By Lemma 4.5, the coefficients are polynomials in the Chern classes of $\mathcal{T}$.

We conclude: if $r$ is such that $i(X_m) = 2^r$, the element $2^r \xi_n(l_0)\xi_n(l_1)\ldots\xi_n(l_{m-1}) \in \mathrm{CH}(\bar{X}_n)$ is a polynomial in the Chern classes of $\mathcal{T}$.

Recall ([6, Theorem 86.12 and Formula 86.5] originally proved in [16]) that the group $\mathrm{CH}(\bar{X}_n)$ is free with a basis given by all $2^n$ products of distinct $\xi_n(l_0), \ldots, \xi_n(l_{n-1})$. The elements $2\xi_n(l_0), \ldots, 2\xi_n(l_{n-1})$ are, up to a sign, the Chern classes of $\mathcal{T}$ ([6, Proposition 86.13]). The additive group of the subring in $\mathrm{CH}(\bar{X}_n)$, generated by these Chern classes, is free with the basis given by the $2^n$ products of distinct $2\xi_n(l_0), \ldots, 2\xi_n(l_{n-1})$. (For the generalization of this fact to all $X_m$ see [9, Theorem 2.1].) Therefore $r = m$. □

We recall two classical facts used in the above proof:

**Lemma 4.4.** *Let $X$ be a smooth variety with a rank $n$ vector bundle $\mathcal{E}$ and let $Y$ be the variety of complete flags in $\mathcal{E}$. Let $x_1, \ldots, x_n$ be the first Chern classes of the tautological line bundles on $Y$. The* $\mathrm{CH}(X)$*-module* $\mathrm{CH}(Y)$ *is free with a basis given by the monomials $x_1^{a_1} \ldots x_n^{a_n}$ satisfying the condition $a_i < i$ for all $i$.*

*Proof.* Viewing $Y \to X$ is a chain of projective bundles, the statement follows from Projective Bundle Theorem [6, Theorem 57.14]. □

**Lemma 4.5.** *The ring of polynomials $\mathbb{Z}[x_1, \ldots, x_n]$ in variables $x_1, \ldots, x_n$, considered as a module over the subring of symmetric polynomials, is free with a basis given by the monomials $x_1^{a_1} \ldots x_n^{a_n}$ satisfying the condition $a_i < i$ for all $i$.*

*Proof.* Apply Lemma 4.4, taking for $X$ the grassmannian of $n$-dimensional subspaces in a vector space of large (better infinite) dimension and taking for $\mathcal{E}$ the tautological vector bundle. □

# Part 2. Even dimension

## 5. Introduction

Let us repeat the introduction of Part 1, replacing the quadratic forms of odd dimension by the even dimensional ones and making the other necessary changes. There are a lot of similarities between the even and the odd dimensional cases; we apologize for repetitions. On the other hand, the picture here is somewhat messier or more complicated in places.

Let $F$ be any field and let $V$ be an $F$-vector space of finite even dimension $2n$ for some integer $n \geq 1$. Let $q\colon V \to F$ be a non-degenerate quadratic form. For any $m = 1, \ldots, n$, the $m$th orthogonal grassmannian $X_m$ of $q$ is defined as the variety of $m$-dimensional totally isotropic subspaces in $V$. Thus, $X_m$ is a closed subvariety inside the usual $m$-grassmannian of $V$. The two extremes here are studied the most in the literature: the projective quadric $X_1$ and the highest orthogonal grassmannian $X_n$.

The variety $X_m$ has a rational point if and only if the Witt index $i_W(q)$ of the quadratic form $q$ is at least $m$. Since $q$ becomes hyperbolic over some multiquadratic field extension, the index $i(X_m)$ of the variety $X_m$ is a power of 2. Namely, $i(X_m)$ is the maximal 2-power dividing the degree of every finite field extension $L/F$ satisfying $i(q_L) \geq m$.

The question considered in this part is as follows: given $n$ and $m$, what is the maximal value of $i(X_m)$ when $F$, $V$, $q$ vary (more precisely, we let $F$ vary over all field extensions of a fixed field) and the discriminant and the Clifford invariant of $q$ (now given by the Brauer class of the total Clifford algebra $C(q)$) are trivial? Since this maximal value is realized at the generic quadratic forms with trivial discriminant and Clifford invariant (defined below), what we want is just to determine $i(X_m)$ in the case of generic $q$. Note that if we drop the condition on triviality of the invariants, the answer to the modified question becomes $2^m$ for any $m$. Clearly, this is an upper bound for the original question.

In the case of trivial invariants, the isomorphism class of the quadratic form $q$ is given by a $\operatorname{Spin}(2n)$-torsor $E$ over $F$. (And any $\operatorname{Spin}(2n)$-torsor over $F$ yields an isomorphism class of a $(2n)$-dimensional quadratic form with trivial discriminant and Clifford invariant. Saying "generic $q$" above we meant that it was given by a *generic torsor* defined as the generic fiber of the quotient morphism $\mathrm{GL}(N) \to \mathrm{GL}(N)/\operatorname{Spin}(2n)$ for an embedding $\operatorname{Spin}(2n) \hookrightarrow \mathrm{GL}(N)$ for some $N$.)

If $m \neq n$, one has $X_m \simeq E/P$ for an appropriate parabolic subgroup $P \subset \operatorname{Spin}(2n+1)$. The variety $X_n$ consists of two connected components each of which is isomorphic to $E/P$. For an arbitrary proper parabolic subgroup $P \subset \operatorname{Spin}(2n)$, the twisted flag variety $E/P$ is either the variety of flags of totally isotropic subspaces in $V$ of some dimensions $1 \leq m_1 < \cdots < m_r \leq n$ (with some $r \geq 1$) or (if and only if $m_r = n$) one of its two isomorphic components. The index of this variety coincides with $i(X_{m_r})$ and therefore does not require additional investigation.

The case of the highest orthogonal grassmannian $X_n$ has been done in [14].[1] (Note that $i(X_n) = i(X_{n-1}) = i(X_{n-2}) = i(X_{n-3})$ if defined.) The answer and proof there appear to be quite complicated. For generic $q$, the integer $i(X_n)$ is equal to $2^t$, where $t$ is approximately $n - 2\log_2(n)$.

For the other extreme – the projective quadric $X_1$, the answer is much more simple: it is just $2 = 2^m$ if we only look at the forms $q$ of dimension at least 12. In other terms, the evident upper bound $i(X_m) \leq 2^m$ turns out to be sharp here. Note that this answer is equivalent to the following assertion: there are anisotropic forms $q$ with trivial discriminant and Clifford invariant in every dimension $\dim q \geq 12$. (We do not know a simple proof for this statement. Several different proofs are presented in the appendix.)

In this part we show for arbitrary $m < n$ that the evident upper bound $2^m$ is sharp provided that $2^{n-m-1} > \dim X_m$ (see Theorem 7.2). This implies that an overwhelming majority of the natural numbers $n$ have the following property: the formula $i(X_m) = 2^m$ holds for $m < t$ and fails for $m > t$. More precisely, the proportion of such $n < N$ tends to 1 when $N \to \infty$.

[1]Due to so-called *exceptional* (in the sense of [3]) isomorphism between $X_n$ and the highest orthogonal grassmannian of any non-degenerate 1-codimensional subform of $q$, our question on the highest orthogonal grassmannian needs not to be considered for quadratic forms of even dimension.

Theorem 7.2 confirms [8, Conjecture 3.3] for $n \geq 14$. For large $n$, the theorem is stronger than the statement of the conjecture.

## 6. Computation of invariants

In notation of §3, let $A'$ be the subgroup of $A$ consisting of the elements with the trivial sum of components.

The orbit of the element $z$ under the action of $A'$ consists of $2^{l-1}$ elements. We write $\check{z} \in R[z]^{A'}$ for the product of the elements in the orbit of $z$. As a warm up, note that the ring $R^{A'}$ is generated by $x_1, \dots, x_m$, the squares $y_1^2, \dots, y_l^2$, and the product $y_1 \dots y_l$.

**Proposition 6.1.** *The $R^{A'}$-algebra $R[z]^{A'}$ is generated by the $l-1$ elements $\check{z}, f_1, \dots, f_{l-2}$.*

*Proof.* Just repeat the proof of Proposition 3.3 (including Lemma 3.2) replacing $A$ by $A'$, $\tilde{z}$ by $\check{z}$, and $l$ by $l-1$. □

## 7. Main result

In this section, $G$ is the split spin group $\mathrm{Spin}(2n)$ for some $n > 1$ over an arbitrary field $F$. As in [11, §8.4], we construct $G$ out of a hyperbolic quadratic form $q$ defined on a $(2n)$-dimensional vector space $V$ with a basis given by vectors $e_i, f_i$, $i = 1, \dots, n$, where $e_i, f_i$ are pairwise orthogonal hyperbolic pairs. Let us fix some $m \in \{1, \dots, n-1\}$, consider the $m$-dimensional totally isotropic subspace generated by $e_1, \dots, e_m$ and let $P \subset G$ be its stabilizer. Then $P$ is a parabolic subgroup in $G$ and the variety $G/P$ is the $m$th orthogonal grassmannian $X_m$ of $q$.

We take for $T \subset P$ the split maximal torus, mapped under the isogeny $G \to G/\mu_2 = \mathrm{O}^+(2n)$ (with the special orthogonal group) to the split maximal torus $T' := \mathbb{G}_\mathrm{m}^n \hookrightarrow \mathrm{O}^+(2n)$ given by $t(e_i) = t_i e_i$ and $t(f_i) = t_i^{-1} f_i$, where $t = (t_1, \dots, t_n) \in \mathbb{G}_\mathrm{m}^n(F)$. We have an exact sequence

$$1 \to \mu_2 \to T \to T' \to 1.$$

Writing $x_1, \dots, x_n$ for the standard basis of $\mathbb{Z}^n = \hat{T}'$, we therefore have $\hat{T} = \hat{T}' + \mathbb{Z}z$, where $z := (x_1 + \cdots + x_n)/2$.

Let us set $l := n - m$. The Weyl group $W$ of $P$ is the direct product of the symmetric group $S_m$ and the Weyl group of $\mathrm{O}^+(2l)$, the latter being a semidirect product of $S_l$ by $A' \subset (\mathbb{Z}/2\mathbb{Z})^l$ for $A'$ introduced in §6. The action of $W$ on $\hat{T}'$ is given by the action of $S_m$ by permutations of $x_1, \dots, x_m$, the action of $S_l$ by permutation of $x_{m+1}, \dots, x_n$, and the action of $A'$ obtained by restriction of the action of $(\mathbb{Z}/2\mathbb{Z})^l$, where the $i$th copy of $\mathbb{Z}/2\mathbb{Z}$ acts by changing the sign of $x_{m+i}$. This action extends uniquely to an action of $W$ on $\hat{T}$.

Let us identify $S(\hat{T}')$ with the ring $R$ from the previous section by identifying $x_{m+i}$ with $y_i$ for $i = 1, \dots, l$. The action of $A'$ we had there is the restriction of the action of $W \supset A'$ we have now. The element $z$ from the previous section corresponds to the element $z$ introduced here and $S(\hat{T}) = R[z]$. The product $\check{z}$ from the previous section becomes an element of $S^{2^{l-1}}(\hat{T})$.

The parabolic subgroup $P$ contains the Borel subgroup $B$ of $G$ defined as the stabilizer of the flag of the totally isotropic subspaces

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \dots, e_n \rangle .$$

We are going to study the image of the composition $\mathsf{S}(\hat{T})^W \hookrightarrow \mathsf{S}(\hat{T}) \to \mathrm{CH}(G/B)$. Note that $G/B$ is the variety of flags of totally isotropic subspaces in $V$ of dimensions $1,\dots,n-1$, which is a component of the variety of flags of totally isotropic subspaces in $V$ of dimensions $1,\dots,n-1,n$. The image of $\mathsf{S}(\hat{T}') \hookrightarrow \mathsf{S}(\hat{T}) \to \mathrm{CH}(G/B)$ is the subring $C \subset \mathrm{CH}(G/B)$ generated by the Chern classes of the tautological line bundles on the latter variety of flags.

**Proposition 7.1.** *The image of $\mathsf{S}(\hat{T})^W$ in $\mathrm{CH}(G/B)$ is contained in the $C$-subalgebra generated by the image of $\check{z} \in \mathsf{S}(\hat{T})$.*

*Proof.* We note that $\mathsf{S}(\hat{T})^W \subset \mathsf{S}(\hat{T})^{A'}$ and apply Proposition 6.1. The Chern classes of the tautological rank $n$ vector bundle on $G/B$ satisfy relations (3.4) (see [6, Formula 86.15]). Besides, $\mathrm{CH}(G/B)$ if (torsion) free. Consequently, the ideal $I \subset R[z] = \mathsf{S}(\hat{T})$ from Lemma 3.5 vanishes in $\mathrm{CH}(G/B)$ and we are done by Lemma 3.5. □

The dimension formula for the variety $X_m$ is as follows:

$$\dim X_m = m(m-1)/2 + 2m(n-m).$$

**Theorem 7.2.** *For $1 \le m < n$, let $q$ be a generic $(2n)$-dimensional quadratic form with trivial discriminant and Clifford invariant and let $X_m$ be its $m$th orthogonal grassmannian. Then $i(X_m) = 2^m$ provided that $2^{n-m-1} > \dim X_m$.*

**Example 7.3.** For $m = 1$, the condition $2^{n-m-1} > \dim X_m$ is satisfied if and only if $\dim q \ge 12$.

*Proof of Theorem 7.2.* Let $X$ be a component of the variety of flags of totally isotropic subspaces of dimensions $1,\dots,n$ for the form $q$. We are going to work with the varieties $X$, $X_1$, $X_m$, $X_n$ and with their base change $\bar{X}$, $\bar{X}_1$, $\bar{X}_m$, $\bar{X}_n$ to the algebraic closure of the base field. We replace $X_n$ by one of its components – the one onto which $X$ projects.

As shown in [17, Statement 2.15], the class $[\mathsf{pt}] \in \mathrm{CH}(\bar{X}_m)$ of a rational point $\mathsf{pt}$ on $\bar{X}_m$ is equal to

$$[\mathsf{pt}] = \xi_m(l_0)\xi_m(l_1)\dots\xi_m(l_{m-1}),$$

where $l_i \in \mathrm{CH}_i(\bar{X}_1)$ is the class of a projective linear $i$-dimensional subspace on $\bar{X}_1$ and $\xi_m$ is the composition

$$\xi_m\colon\ \mathrm{CH}(\bar{X}_1) \to \mathrm{CH}(\bar{X}_{1,m}) \to \mathrm{CH}(\bar{X}_m)$$

of the pull-back followed by the push-forward with respect to the projections of the flag variety $X_{1,m} \subset X_1 \times X_m$.

The image of $[\mathsf{pt}]$ under the pull-back $\mathrm{CH}(\bar{X}_m) \to \mathrm{CH}(\bar{X})$ can be computed via [17, Lemma 2.6], a statement similar to Lemma 4.3. What we get is a polynomial in $x_{m+1},\dots,x_n$ (the first Chern classes of the corresponding tautological line bundles on $X$) with coefficients in $\mathrm{CH}(\bar{X}_n)$ such that the coefficient at $x_{m+1}^m x_{m+2}^{m+1}\dots x_n^{n-1}$ is

$$\xi_n(l_0)\xi_n(l_1)\dots\xi_n(l_{m-1}) \in \mathrm{CH}(\bar{X}_n).$$

By Proposition 7.1, since $\check{z} \in \mathsf{S}^{2^{n-m-1}}(\hat{T})$ and $2^{n-m-1} > \dim X_m$, every element $a$ in the image of the composition

$$\mathrm{CH}(X_m) \to \mathrm{CH}(\bar{X}_m) \to \mathrm{CH}(\bar{X})$$

is an integral polynomial in the first Chern classes $x_1, \dots, x_n$ of the tautological line bundles on $X$. By Lemma 4.4, the element $a$ can be written uniquely as a polynomial in $x_1, \dots, x_n$ with coefficients in $\mathrm{CH}(\bar{X}_n)$, where each $x_i$ appears only in degrees $< i$. By Lemma 4.5, the coefficients are polynomials in the Chern classes of $\mathcal{T}$.

We conclude: if $r$ is such that $i(X_m) = 2^r$, the element $2^r \xi_n(l_0)\xi_n(l_1)\dots\xi_n(l_{m-1}) \in \mathrm{CH}(\bar{X}_n)$ is a polynomial in the Chern classes of $\mathcal{T}$.

Recall ([6, Theorem 86.12 and Formula 86.5] originally proved in [16]) that the group $\mathrm{CH}(\bar{X}_n)$ is free with a basis given by all $2^{n-1}$ products of distinct $\xi_n(l_0), \dots, \xi_n(l_{n-2})$. The elements $2\xi_n(l_0), \dots, 2\xi_n(l_{n-2})$ are, up to a sign, the first $n-1$ Chern classes of $\mathcal{T}$ ([6, Proposition 86.13]), the $n$th Chern class being 0 (see [6, Proposition 86.17] or [9, Theorem 2.1]). The additive group of the subring in $\mathrm{CH}(\bar{X}_n)$, generated by these Chern classes, is free with the basis given by the $2^{n-1}$ products of distinct $2\xi_n(l_0), \dots, 2\xi_n(l_{n-2})$. (For the generalization of this fact to all $X_m$ see [9, Theorem 2.1].) Therefore $r = m$. □

## Appendix. Quadrics

In this appendix we list several different proofs of the fact $\Phi$ that $i(X_1) = 2$ for a generic quadratic form $q$ of sufficiently large even dimension with trivial discriminant and Clifford invariant. Note that by taking a 1-codimensional subform in $q$, the statement $\Phi$ implies the similar statement on the quadratic forms of odd dimension.

**Steenrod operations:** $\Phi$ is a consequence of [6, Proposition 82.7] (which is due to A. Vishik), whose proof makes use of Steenrod operations on the modulo 2 Chow groups of algebraic varieties. In particular, it became available in characteristic 2 only after the recent [12].

More precisely, the Steenrod operations are used in the proof of [6, Corollary 80.8] on the possible size of binary correspondences.

**Essential dimension:** In characteristic 0, $\Phi$ follows from [2, Theorem 4.2]. The theorem roughly states that a generic quadratic form with trivial discriminant and Clifford invariant and of sufficiently large dimension contains no proper even-dimensional subforms of trivial discriminant. A non-degenerate quadratic form is anisotropic if and only if it contains no 2-dimensional subforms of trivial discriminant (i.e., hyperbolic planes).

**Totaro's torsion index:** $\Phi$ can be deduced from the computation of the index of the highest orthogonal grassmannian made in [14]. Indeed, if for some $n \geq 6$, a generic $2n$-dimensional quadratic form $q$ with trivial discriminant and Clifford invariant is isotropic, then $l_0 \in \mathrm{CH}_0(\bar{X}_1)$ is in the image of $\mathrm{CH}(X_1) \to \mathrm{CH}(\bar{X}_1)$ so that $\xi_n(l_0) \in \mathrm{CH}(\bar{X}_n)$ is in the image of $\mathrm{CH}(X_n) \to \mathrm{CH}(\bar{X}_n)$. The latter image modulo 2 is known to be the subring generated by $\xi_n(l_{n-2})$. By [6, Formula (86.15)], it follows that $n-1$ is a 2-power.

On the other hand, since $q$ is isotropic, the index $i(X_n)$, computed in [14], can't be higher than the index $i(X'_{n-1})$ of the highest orthogonal grassmannian $X'_{n-1}$ of a generic $2(n-1)$-dimensional quadratic form $q'$ with trivial discriminant and Clifford invariant. It follows by [14, Theorem 0.1] that $n-1$ is not a 2-power.

**Our main result:** $\Phi$ is a particular case of Theorem 7.2 (see Example 7.3).

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