# KRULL DIMENSION OF THE NEGLIGIBLE QUOTIENT IN MOD $p$ COHOMOLOGY OF A FINITE GROUP 

M. GHERMAN AND A. MERKURJEV


#### Abstract

For a finite group $G$, a $G$-module $M$, and a field $F$, an element $u \in H^{d}(G, M)$ is negligible over $F$ if for each field extension $L / F$ and every continuous group homomorphism from $\operatorname{Gal}\left(L_{\text {sep }} / L\right)$ to $G, u$ belongs to the kernel of the induced homomorphism $H^{d}(G, M) \rightarrow$ $H^{d}(L, M)$. For $p$ a prime and a trivial $G$-action on the coefficients, the negligible elements in the cohomology ring $H^{*}(G, \mathbb{Z} / p \mathbb{Z})$ form an ideal. We show that when $p$ is odd or $p=2$ and either $|G|$ is odd or $F$ is not formally real, the Krull dimension of the quotient of mod $p$ cohomology by the negligible ideal is 0 . However, when $p=2,|G|$ is even, and $F$ is formally real, the Krull dimension of the quotient of $\bmod 2$ cohomology of a finite 2 -group by the negligible ideal is 1 . We further compute the generators of the negligible ideal in the $\bmod p$ cohomology of elementary abelian $p$-groups.


## 1. Introduction

The notion of negligible cohomology was introduced by J.-P. Serre in [17] (see also [7, Part I § 26]). Let $G$ be a finite group, $M$ a $G$-module, and $F$ a field. A continuous group homomorphism $j: \Gamma_{L}=\operatorname{Gal}\left(L_{\text {sep }} / L\right) \rightarrow G$ from the absolute Galois group $\Gamma_{L}$ of a field extension $L$ of $F$ to $G$ yields a homomorphism $j^{*}: H^{d}(G, M) \rightarrow H^{d}(L, M)$ of cohomology groups for every $d \geq 0$. An element $u \in H^{d}(G, M)$ is negligible over $F$ if $u \in \operatorname{ker}\left(j_{*}\right)$ for all field extensions $L / F$ and all $j$.

A fundamental and difficult problem in Galois theory is to characterize those profinite groups which are realizable as absolute Galois groups of fields. One of the most common approaches has been to find constraints on the cohomology of absolute Galois groups. For instance, the Bloch-Kato conjecture, proved by Rost and Voevodsky, provides a presentation of the cohomology of absolute Galois groups with generators in degree 1 and relations in degree 2. In the present paper, we make precise the notion that most classes in the cohomology of a finite group disappear when mapped to the cohomology of an absolute Galois group. We can interpret the size of negligible cohomology as a further restriction on profinite groups that are realizable as absolute Galois groups of fields.

For a prime $p$, we will assume throughout this paper that $M=\mathbb{Z} / p \mathbb{Z}$ has a trivial $G$ action. If $\operatorname{char}(F)=p>0$, then $H^{d}(K, \mathbb{Z} / p \mathbb{Z})=0$ for $d \geq 2$ by [16, Chapter II Proposition 3] so $H^{d}(G, \mathbb{Z} / p \mathbb{Z})$ is entirely negligible for $d \geq 2$. We will, therefore, assume $F$ is a field with $\operatorname{char}(F) \neq p$ when computing the negligible classes of $H^{*}(G, \mathbb{Z} / p \mathbb{Z})$. The negligible classes of $H^{d}(G, \mathbb{Z} / p \mathbb{Z})$ over $F$ are the same as that over $F\left(\xi_{p}\right)$ for $\xi_{p}$ a primitive $p$ th root

[^0]of unity by [8, Corollary 2.4]. Hence, we will assume that $F$ contains a primitive $p$ th root of unity $\xi_{p}$. In order to discuss Krull dimension, we define the commutative ring
\[

\mathcal{H}(G, \mathbb{Z} / p \mathbb{Z})= $$
\begin{cases}H^{\text {even }}(G, \mathbb{Z} / p \mathbb{Z}) & \text { if } p \neq 2 \\ H^{*}(G, \mathbb{Z} / 2 \mathbb{Z}) & \text { if } p=2\end{cases}
$$
\]

Since inflation maps are ring homomorphisms, the negligible elements of the ring $\mathcal{H}(G, \mathbb{Z} / p \mathbb{Z})$ form an ideal, denoted $\mathcal{I}(G, \mathbb{Z} / p \mathbb{Z})$. We write $Q(G, \mathbb{Z} / p \mathbb{Z})=\mathcal{H}(G, \mathbb{Z} / p \mathbb{Z}) / \mathcal{I}(G, \mathbb{Z} / p \mathbb{Z})$ for the negligible quotient. In the possibly non-commutative $\operatorname{ring} H^{*}(G, \mathbb{Z} / p \mathbb{Z})$, we denote the two-sided ideal of negligible elements $I(G, \mathbb{Z} / p \mathbb{Z})$.

The level of a field $F$, denoted $s(F)$, is the least number of squares that sum to -1 in $F$. If -1 cannot be written as a sum of squares, then $F$ is formally real. Pfister's Level Theorem, [12, Theorem 2.2], proves that when $s(F)$ is finite, $s(F)$ is a power of 2 . If $F$ is a field with $s(F)=2^{r}$, we can, equivalently, say that the $r$-fold Pfister form $\langle\langle 1, \ldots, 1\rangle\rangle$ is anisotropic over $F$ while the $(r+1)$-fold Pfister form $\langle\langle 1, \ldots, 1,1\rangle\rangle$ is isotropic over $F$. By [4, Section 16], the class $(-1)^{r+1} \in H^{*}(F, \mathbb{Z} / p \mathbb{Z})$ is trivial while $(-1)^{r} \in H^{*}(F, \mathbb{Z} / p \mathbb{Z})$ is not. For a proof of the result, see [13, Theorem 4.1].

Quillen proved in [15, Corollary 7.8] that the Krull dimension of $\mathcal{H}^{*}(G, \mathbb{Z} / p \mathbb{Z})$ is equal to the maximum rank of an elementary abelian $p$-subgroup of $G$. With Quillen's result as inspiration, we prove the following two theorems about the Krull dimension of the negligible quotient.
Theorem (2.2). Let $p$ be a prime, $G$ a finite group, and $F$ a field. If $p=2$, assume that $F$ is not formally real or $G$ has odd order. Then the negligible quotient $Q(G, \mathbb{Z} / p \mathbb{Z})$ is finite. In particular, $Q(G, \mathbb{Z} / p \mathbb{Z})$ has Krull dimension 0 .

Theorem (3.3). Let $G$ be a finite group of even order and $F$ a formally real field. Then the negligible quotient $Q(G, \mathbb{Z} / 2 \mathbb{Z})$ has Krull dimension 1.

In [14], Quillen and Venkov proved that nilpotent elements in the group cohomology of a finite group $G$ are detected on the elementary abelian $p$-subgroups of $G$. Likewise, the elementary abelian $p$-subgroups of $G$ provide an effective tool for detecting negligible classes in the cohomology of $G$. We conclude the paper with a computation of the $\bmod p$ negligible cohomology ideal of elementary abelian $p$-groups in Section 4 .
1.1. Notation and Facts. Let $K$ be a field extension of $F$. We will fix a primitive $p$ th root of unity $\xi_{p} \in F$ throughout the paper. We identify the $p$ th roots of unity $\mu_{p} \subset K$ with $\mathbb{Z} / p \mathbb{Z}$. Then $H^{1}(K, \mathbb{Z} / p \mathbb{Z}) \simeq H^{1}\left(K, \mu_{p}\right) \simeq K^{\times} /\left(K^{\times}\right)^{p}$, and we write an element of $H^{1}(K, \mathbb{Z} / p \mathbb{Z})$ as a class $(a)$ for $a\left(K^{\times}\right)^{p} \in K^{\times} /\left(K^{\times}\right)^{p}$. Let $\left(a_{i}\right) \in K^{\times} /\left(K^{\times}\right)^{p}$ for $1 \leq i \leq d$. We often write $\left(a_{1}, \ldots, a_{d}\right)$ for the cup product $\left(a_{1}\right) \cup \cdots \cup\left(a_{d}\right)$ in $H^{d}(K, \mathbb{Z} / p \mathbb{Z})$. Note that $(a, a)=(a,-1)$ and $(a, b)+(b, a)=0$ for all $(a),(b) \in H^{1}(K, \mathbb{Z} / p \mathbb{Z})$.

The Norm Residue Isomorphism Theorem [10] reveals that the ideal $H^{>0}(K, \mathbb{Z} / p \mathbb{Z})$ is generated by elements of $H^{1}(K, \mathbb{Z} / p \mathbb{Z})$. Therefore, it is often sufficient to check properties on generators $(a) \in H^{1}(K, \mathbb{Z} / p \mathbb{Z})$ of $H^{>0}(K, \mathbb{Z} / p \mathbb{Z})$.

When $p=2$, let $\left(a_{i}\right) \in H^{1}(K, \mathbb{Z} / 2 \mathbb{Z})$ for $1 \leq i \leq d$. Since $H^{*}(K, \mathbb{Z} / 2 \mathbb{Z})$ is a commutative ring, $\left(a_{1}, \ldots, a_{d}\right)^{2}=\left(a_{1}, \ldots, a_{d}\right) \cup(-1)^{d}$. The squaring map and cup product by $(-1)^{d}$ are linear. Therefore, $\alpha^{2}=\alpha \cup(-1)^{d}$ for any $\alpha \in H^{d}(K, \mathbb{Z} / 2 \mathbb{Z})$. An inductive argument reveals $\alpha^{k+1}=\alpha \cup(-1)^{d k}$.

## 2. KRULL DIMENSION OF THE NEGLIGIBLE QUOTIENT OVER FIELDS THAT ARE NOT FORMALLY REAL

In all cases except when $p=2, F$ is formally real, and $G$ has even order, we prove that the mod $p$ cohomology of a finite group $G$ becomes entirely negligible after some degree. We begin with a more general result about the nilpotence of elements in Galois cohomology.

Lemma 2.1. Let $p$ be a prime. If $p=2$, assume that the field $K$ is not formally real. Then every element of $H^{>0}(K, \mathbb{Z} / p \mathbb{Z})$ is nilpotent.

Proof. In the graded ring $H^{*}(K, \mathbb{Z} / p \mathbb{Z})$, the sum of nilpotent elements is nilpotent and the $p$ th power map is linear. It is thus sufficient to check nilpotence on homogeneous generators $(a) \in H^{1}(K, \mathbb{Z} / p \mathbb{Z})$ of $H^{>0}(K, \mathbb{Z} / p \mathbb{Z})$. When $p$ is odd, we have $(a)^{2}=(a,-1)=0$. When $p=2$ and $s(K)=2^{r},(-1)^{r+1}$ is trivial in $H^{r+1}(K, \mathbb{Z} / 2 \mathbb{Z})$. Let $m$ be a power of 2 such that $r+1 \leq m-1$. Then $(a)^{m}=(a) \cup(-1)^{m-1}=0$.

Theorem 2.2. Let $p$ be a prime, $G$ a finite group, and $F$ a field. If $p=2$, assume that $F$ is not formally real or $G$ has odd order. Then the negligible quotient $Q(G, \mathbb{Z} / p \mathbb{Z})$ is finite. In particular, $Q(G, \mathbb{Z} / p \mathbb{Z})$ has Krull dimension 0 .

Proof. If $p=2$ and $|G|$ is odd, $H^{>0}(G, \mathbb{Z} / 2 \mathbb{Z})=0$. Hence, we may assume that $F$ is not formally real when $p=2$. By [6, Corollary 7.4.6], the $\operatorname{ring} \mathcal{H}(G, \mathbb{Z} / p \mathbb{Z})$ is finitely generated. Let $K$ be a field extension of $F$ and $j: \Gamma_{K} \rightarrow G$ a continuous group homomorphism. The image of each generator via $j^{*}$ will be nilpotent in $H^{>0}(K, \mathbb{Z} / p \mathbb{Z})$ by Lemma 2.1. Therefore, each generator of $\mathcal{H}(G, \mathbb{Z} / p \mathbb{Z})$ is in the radical of $\mathcal{I}(G, \mathbb{Z} / p \mathbb{Z})$ and, hence, $Q(G, \mathbb{Z} / p \mathbb{Z})$ is finite. We conclude that $Q(G, \mathbb{Z} / p \mathbb{Z})$ is a ring of Krull dimension 0 .

## 3. Krull dimension of the negligible quotient over formally real fields

The final case to consider is when $p=2, F$ is formally real, and $G$ has even order. With these assumptions, we prove the Krull dimension of the negligible quotient is always 1.

Lemma 3.1. Let $G$ be a finite group of even order. Assume that $F$ is formally real. Then the Krull dimension of $Q(G, \mathbb{Z} / 2 \mathbb{Z})$ is positive.

Proof. Let $H$ be an order 2 cyclic subgroup of $G$. By [8, Proposition 2.3(1)], the restriction

$$
\text { res : } \mathcal{H}(G, \mathbb{Z} / 2 \mathbb{Z}) \rightarrow \mathcal{H}(H, \mathbb{Z} / 2 \mathbb{Z})
$$

factors as $f: Q(G, \mathbb{Z} / 2 \mathbb{Z}) \rightarrow Q(H, \mathbb{Z} / 2 \mathbb{Z})$. By [5, Theorem 7.1], $\mathcal{H}(H, \mathbb{Z} / 2 \mathbb{Z})$ is a finite algebra over the subring $\operatorname{im}($ res $)$ so $Q(H, \mathbb{Z} / 2 \mathbb{Z})$ is a finite algebra over the subring $\operatorname{im}(f)$. [1, Corollary 5.9] shows $\operatorname{dim}(Q(G, \mathbb{Z} / 2 \mathbb{Z})) \geq \operatorname{dim}(Q(H, \mathbb{Z} / 2 \mathbb{Z}))$. By Theorem 4.1,

$$
Q(H, \mathbb{Z} / 2 \mathbb{Z})=\mathcal{H}(H, \mathbb{Z} / 2 \mathbb{Z}) \simeq \mathbb{Z} / 2 \mathbb{Z}[x]
$$

has Krull dimension 1.
Lemma 3.2. Let $G$ be a finite group. The Krull dimension of $Q(G, \mathbb{Z} / 2 \mathbb{Z})$ is at most 1 .
Proof. Let $u$ and $v$ be homogeneous elements of $H^{*}(G, \mathbb{Z} / 2 \mathbb{Z})$. Denote $k=\operatorname{deg}(u)$ and $\ell=\operatorname{deg}(v)$. We will show that $u v\left(u^{\ell}+v^{k}\right)$ is negligible. Let $K$ be a field extension of $F$
and $j: \Gamma_{K} \rightarrow G$ a continuous group homomorphism. Let $\alpha=j^{*}(u)$ and $\beta=j^{*}(v)$. Then

$$
\begin{aligned}
j^{*}\left(u^{\ell+1} v+u v^{k+1}\right) & =\alpha^{\ell+1} \cup \beta+\alpha \cup \beta^{k+1} \\
& =\alpha \cup(-1)^{k \ell} \cup \beta+\alpha \cup \beta \cup(-1)^{k \ell} \\
& =(\alpha \cup \beta+\alpha \cup \beta) \cup(-1)^{k \ell} \\
& =0
\end{aligned}
$$

by Section 1.1. We conclude that elements of the form $u v\left(u^{\ell}+v^{k}\right)$ are negligible.
For a set of generators $\left\{u_{1}, \ldots, u_{m}\right\}$ of $R=H^{*}(G, \mathbb{Z} / 2 \mathbb{Z})$ with $d_{i}=\operatorname{deg}\left(u_{i}\right)$, define the ideal $I=\left\langle u_{i} u_{j}\left(u_{i}^{d_{j}}+u_{j}^{d_{i}}\right): 1 \leq i<j \leq m\right\rangle$. We showed above that $I \subset I(G, \mathbb{Z} / 2 \mathbb{Z})$. Let $P$ be a prime ideal of $R$ that contains $I(G, \mathbb{Z} / 2 \mathbb{Z})$ and, thus, $I$. It suffices to show that $\operatorname{dim}(R / P) \leq 1$ since

$$
\operatorname{dim}(Q(G, \mathbb{Z} / 2 \mathbb{Z}))=\max _{P \supset I(G, \mathbb{Z} / 2 \mathbb{Z})} \operatorname{dim}(R / P)
$$

If $u_{i} \in P$ for all $1 \leq i \leq m$, then $R / P=\mathbb{Z} / 2 \mathbb{Z}$ and $\operatorname{dim}(R / P)=0$. We may assume that $u_{i} \notin P$ for some $1 \leq i \leq m$. Since $u_{i} u_{j}\left(u_{i}^{d_{j}}+u_{j}^{d_{i}}\right) \in P$ and $P$ is prime, $u_{j} \in P$ or $u_{i}^{d_{j}}+u_{j}^{d_{i}} \in P$ for every $j$. For the ring homomorphism $\varphi: \mathbb{Z} / 2 \mathbb{Z}[t] \rightarrow R / P$ defined as $\varphi(t)=u_{i}, u_{j}$ is integral over $\operatorname{im}(\varphi)$ in either case. Thus $R / P$ is a finite $\mathbb{Z} / 2 \mathbb{Z}[t]$-algebra so $\operatorname{dim}(R / P) \leq \operatorname{dim}(\mathbb{Z} / 2 \mathbb{Z}[t])=1$.

Theorem 3.3. Let $G$ be a finite group of even order and $F$ a formally real field. Then the negligible quotient $Q(G, \mathbb{Z} / 2 \mathbb{Z})$ has Krull dimension 1.

## 4. Negligible cohomology ideal of elementary abelian p-Groups

In this section, $G$ is an elementary abelian $p$-group of rank $n$. We wish to compute the negligible cohomology ideal of the mod $p$ cohomology of $G$. We will first study the $p=2$ case, which is a generalization of Serre's computation of negligible classes over $\mathbb{Q}$ for elementary abelian 2-groups found in [7, Lemma 26.4]. By [2, Proposition 4.5.4], the mod 2 cohomology of a rank $n$ elementary abelian 2-group $G$ is a polynomial ring in $n$ variables,

$$
H^{*}(G, \mathbb{Z} / 2 \mathbb{Z}) \simeq \mathbb{Z} / 2 \mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]
$$

where $\left\{x_{1}, \ldots, x_{n}\right\}$ is a basis for $H^{1}(G, \mathbb{Z} / 2 \mathbb{Z})$ as a $\mathbb{Z} / 2 \mathbb{Z}$-vector space.
Throughout this section, we will denote $\{1,2, \ldots, n\}$ by $[1, n]$.
Theorem 4.1. Let $G$ be an elementary abelian 2-group of rank $n$ and $F$ a field with $\operatorname{char}(F) \neq 2$.
(a) If $F$ is formally real, then $I(G, \mathbb{Z} / 2 \mathbb{Z})$ over $F$ is generated by

$$
\left\{x_{i} x_{j}^{2}+x_{j} x_{i}^{2}: 1 \leq i<j \leq n\right\}
$$

(b) If $s(F)=2^{r}>1$, then $I(G, \mathbb{Z} / 2 \mathbb{Z})$ over $F$ is generated by

$$
\left\{x_{i} x_{j}^{2}+x_{j} x_{i}^{2}: 1 \leq i<j \leq n\right\} \cup\left\{x_{i}^{r+2}: 1 \leq i \leq n\right\} .
$$

(c) If $s(F)=1$, then $I(G, \mathbb{Z} / 2 \mathbb{Z})$ over $F$ is generated by

$$
\left\{x_{i}^{2}: 1 \leq i \leq n\right\} .
$$

Proof. Let $I$ be the ideal generated by the elements in the proposition statement for an elementary abelian 2-group of rank $n$. We will first prove that $I \subset I(G, \mathbb{Z} / 2 \mathbb{Z})$. Let $K$ be a field extension of $F$ and $j: \Gamma_{K} \rightarrow G$ be a continuous group homomorphism. Denote $\left(a_{i}\right)=j^{*}\left(x_{i}\right) \in H^{1}(K, \mathbb{Z} / 2 \mathbb{Z}) \simeq K^{\times} /\left(K^{\times}\right)^{2}$.

$$
j^{*}\left(x_{i} x_{j}^{2}+x_{j} x_{i}^{2}\right)=\left(a_{i}, a_{j}, a_{j}\right)+\left(a_{j}, a_{i}, a_{i}\right)=\left(a_{i}, a_{j},-1\right)+\left(a_{j}, a_{i},-1\right)=0
$$

If $s(F)=2^{r}$, then $(-1)^{r+1}$ is trivial and

$$
j^{*}\left(x_{i}^{r+2}\right)=\left(a_{i}\right)^{r+2}=\left(a_{i}\right) \cup(-1)^{r+1}=0 .
$$

We will now show that $I(G, \mathbb{Z} / 2 \mathbb{Z}) \subset I$. Define the iterated Laurent series field $E=$ $F\left(\left(a_{1}\right)\right)\left(\left(a_{2}\right)\right) \cdots\left(\left(a_{n}\right)\right)$ with indeterminates $a_{i}$. For $S \subset[1, n]$, denote $x_{S}=\prod_{i \in S} x_{i}$ and $\left(a_{S}\right)=\prod_{i \in S}\left(a_{i}\right)$ in $H^{|S|}(K, \mathbb{Z} / 2 \mathbb{Z})$. Then $H^{*}(E, \mathbb{Z} / 2 \mathbb{Z})$ is a free $H^{*}(F, \mathbb{Z} / 2 \mathbb{Z})$-module with basis $\left\{\left(a_{S}\right): S \subset[1, n]\right\}$ by [11, Theorem 3]. The field extension $E\left(\sqrt{a_{1}}, \ldots, \sqrt{a_{n}}\right)$ over $E$ is Galois with Galois group $G$ acting by $g \cdot \sqrt{a_{i}}=(-1)^{x_{i}(g)} \sqrt{a_{i}}$ for $g \in G$. As a result, there is a continuous group homomorphism $j_{E}: \Gamma_{E} \rightarrow G$, which induces a ring homomorphism $j_{E}^{*}: H^{*}(G, \mathbb{Z} / 2 \mathbb{Z}) \rightarrow H^{*}(E, \mathbb{Z} / 2 \mathbb{Z})$.

Define the subset $T=\left\{x_{S} x_{j}^{i}: S \subset[1, n], j \in S\right.$ maximal, $\left.0 \leq i\right\}$ of $H^{*}(G, \mathbb{Z} / 2 \mathbb{Z})$ if $F$ is formally real or $T=\left\{x_{S} x_{j}^{i}: S \subset[1, n], j \in S\right.$ maximal, $\left.0 \leq i \leq r+2\right\}$ if $F$ is not formally real and $s(F)=2^{r}$. Denote by $W$ the subspace of $H^{*}(G, \mathbb{Z} / 2 \mathbb{Z})$ generated by $T$. Note that, modulo $I$, every element of $H^{*}(G, \mathbb{Z} / 2 \mathbb{Z})$ may be reduced to an element of $W$. Further, for all $x_{S} x_{j}^{i} \in T$,

$$
j_{E}^{*}\left(x_{S} x_{j}^{i}\right)=\left(a_{S}\right) \cup\left(a_{j}\right)^{i}=\left(a_{S}\right) \cup(-1)^{i} .
$$

Since $\left\{\left(a_{S}\right): S \subset[1, n]\right\}$ is linearly independent in $H^{*}(E, \mathbb{Z} / 2 \mathbb{Z})$ as a $H^{*}(F, \mathbb{Z} / 2 \mathbb{Z})$-module, the restriction of $j_{E}^{*}$ to $W$ is injective. We build the following commutative square.


A diagram chase implies that $f$ is injective and $I(G, \mathbb{Z} / 2 \mathbb{Z}) \subset I$.
We will now focus on the case when $p$ is an odd prime. By [2, Proposition 4.5.4], the mod $p$ cohomology of an elementary abelian $p$-group $G$ is a polynomial ring over the exterior algebra of the character group $G^{*}$ of $G$,

$$
H^{*}(G, \mathbb{Z} / p \mathbb{Z}) \simeq \Lambda\left(G^{*}\right)\left[y_{1}, \ldots, y_{n}\right]
$$

where $\operatorname{deg}\left(y_{j}\right)=2$. Let $\left\{x_{1}, \ldots, x_{n}\right\}$ be a basis for $H^{1}(G, \mathbb{Z} / 2 \mathbb{Z})$ as a $\mathbb{Z} / 2 \mathbb{Z}$-vector space. For each $1 \leq i \leq n, y_{i}=B\left(x_{i}\right)$ for $B: H^{1}(G, \mathbb{Z} / p \mathbb{Z}) \rightarrow H^{2}(G, \mathbb{Z} / p \mathbb{Z})$ the Bockstein homomorphism.

The following result can be found in [3, proof of Proposition 3.2].
Lemma 4.2. Let $K$ be a field and $\alpha \in H^{1}(K, \mathbb{Z} / p \mathbb{Z})$. Then $B(\alpha)=\alpha \cup\left(\xi_{p}\right)$.

Proof. Let $\tilde{B}: H^{1}(K, \mathbb{Z} / p \mathbb{Z}) \rightarrow H^{2}(K, \mathbb{Z})$ denote the integral Bockstein homomorphism. The homomorphism $f: \mathbb{Z} \rightarrow K_{\text {sep }}^{\times}$satisfying $f(1)=\xi_{p}$ factors through $\mathbb{Z} / p \mathbb{Z}$. We can build the following commutative diagram.


By [9, Proposition 4.7.3, Corollary 2.5.5, and Proposition 4.7.1], $f^{*}(\tilde{B}(\alpha))$ is $\alpha \cup\left(\xi_{p}\right)$ in $H^{2}(K, \mathbb{Z} / p \mathbb{Z})$. Therefore, $B(\alpha)=\alpha \cup\left(\xi_{p}\right)$ by commutativity.

Let $K$ be a field extension of $F$ and $j: \Gamma_{K} \rightarrow G$ be a continuous group homomorphism. The Bockstein commutes with inflation so, for $x \in H^{1}(G, \mathbb{Z} / p \mathbb{Z})$,

$$
\begin{equation*}
j^{*}(B(x))=B\left(j^{*}(x)\right)=j^{*}(x) \cup\left(\xi_{p}\right) \tag{1}
\end{equation*}
$$

by Lemma 4.2 .
Theorem 4.3. Let $p$ be an odd prime. Let $G$ be an elementary abelian $p$-group of rank $n$.
(a) If $F$ does not contain a primitive $p^{2}$ root of unity, then $I(G, \mathbb{Z} / p \mathbb{Z})$ over $F$ is generated by

$$
\left\{x_{i} y_{j}+x_{j} y_{i}: 1 \leq i \leq j \leq n\right\} \cup\left\{y_{i} y_{j}: 1 \leq i \leq j \leq n\right\}
$$

(b) If $F$ contains a primitive $p^{2}$ root of unity, then $I(G, \mathbb{Z} / p \mathbb{Z})$ over $F$ is generated by

$$
\left\{y_{i}: 1 \leq i \leq n\right\}
$$

Proof. Let $I$ be the ideal generated by the elements in the proposition statement for an elementary abelian $p$-group of rank $n$. We will first prove that $I \subset I(G, \mathbb{Z} / p \mathbb{Z})$. Let $K$ be a field extension of $F$ and $j: \Gamma_{K} \rightarrow G$ be a continuous group homomorphism. Denote $\left(a_{i}\right)=j^{*}\left(x_{i}\right) \in H^{1}(K, \mathbb{Z} / p \mathbb{Z}) \simeq K^{\times} /\left(K^{\times}\right)^{p}$ so

$$
j^{*}\left(y_{i}\right)=j^{*}\left(B\left(x_{i}\right)\right)=B\left(j^{*}\left(x_{i}\right)\right)=B\left(a_{i}\right)=\left(a_{i}, \xi_{p}\right)
$$

by equation (1). We have

$$
\begin{aligned}
j^{*}\left(x_{i} y_{j}+x_{j} y_{i}\right) & =\left(a_{i}, a_{j}, \xi_{p}\right)+\left(a_{j}, a_{i}, \xi_{p}\right)=0 \\
j^{*}\left(y_{i} y_{j}\right) & =\left(a_{i}, \xi_{p}, a_{j}, \xi_{p}\right)=-\left(a_{i}, a_{j}, \xi_{p}, \xi_{p}\right)=-\left(a_{i}, a_{j}, \xi_{p},-1\right)=0
\end{aligned}
$$

If $F$ contains a primitive $p^{2}$ root of unity $\xi_{p^{2}}$, we obtain

$$
j^{*}\left(y_{i}\right)=\left(a_{i}, \xi_{p}\right)=\left(a_{i}, \xi_{p^{2}}^{p}\right)=p\left(a_{i}, \xi_{p^{2}}\right)=0 .
$$

We will now show that $I(G, \mathbb{Z} / p \mathbb{Z}) \subset I$. Define the field extension $E$ of $F$ as in the proof of Theorem 4.1. Once again, by [11, Theorem 3], $H^{*}(E, \mathbb{Z} / p \mathbb{Z})$ is a free $H^{*}(F, \mathbb{Z} / p \mathbb{Z})$-module with basis $\left\{\left(a_{S}\right): S \subset[1, n]\right\}$. As before, we have $j_{E}^{*}: H^{*}(G, \mathbb{Z} / p \mathbb{Z}) \rightarrow H^{*}(E, \mathbb{Z} / p \mathbb{Z})$.

Define the subsets

$$
\begin{aligned}
& T_{1}=\left\{x_{S}: S \subset[1, n]\right\} \\
& T_{2}=\left\{x_{S} y_{j}: S \subset[1, n], i<j \text { for each } i \in S\right\}
\end{aligned}
$$

of $H^{*}(G, \mathbb{Z} / p \mathbb{Z})$. If $F$ does not contain a $p^{2}$ root of unity, let $T=T_{1} \cup T_{2}$. If $F$ does contain a $p^{2}$ root of unity, let $T=T_{1}$. Denote by $W$ the subspace of $H^{*}(G, \mathbb{Z} / p \mathbb{Z})$ generated by $T$.

Note that, modulo $I$, every element of $H^{*}(G, \mathbb{Z} / p \mathbb{Z})$ may be reduced to an element of $W$. Further,

$$
\begin{aligned}
j_{E}^{*}\left(x_{S}\right) & =\left(a_{S}\right) \\
j_{E}^{*}\left(x_{S} y_{j}\right) & =\left(a_{S}\right) \cup\left(a_{j}, \xi_{p}\right)=\left(a_{S \cup\{j\}}\right) \cup\left(\xi_{p}\right) .
\end{aligned}
$$

Since $\left\{\left(a_{S}\right): S \subset[1, n]\right\}$ is linearly independent in $H^{*}(E, \mathbb{Z} / p \mathbb{Z})$ as a $H^{*}(F, \mathbb{Z} / p \mathbb{Z})$-module, the restriction of $j_{E}^{*}$ to $W$ is injective. We build the following commutative square.


A diagram chase implies that $f$ is injective and $I(G, \mathbb{Z} / p \mathbb{Z}) \subset I$.

## References

[1] M. F. Atiyah and I. G. Macdonald, Introduction to commutative algebra, economy ed., Addison-Wesley Series in Mathematics, Westview Press, Boulder, CO, 2016, For the 1969 original see [ MR0242802]. MR 3525784
[2] Jon F. Carlson, Lisa Townsley, Luis Valeri-Elizondo, and Mucheng Zhang, Cohomology rings of finite groups, Algebra and Applications, vol. 3, Kluwer Academic Publishers, Dordrecht, 2003, With an appendix: Calculations of cohomology rings of groups of order dividing 64 by Carlson, Valeri-Elizondo and Zhang. MR 2028960
[3] Ido Efrat and Ján Mináč, On the descending central sequence of absolute Galois groups, Amer. J. Math. 133 (2011), no. 6, 1503-1532. MR 2863369
[4] Richard Elman, Nikita Karpenko, and Alexander Merkurjev, The algebraic and geometric theory of quadratic forms, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, Providence, RI, 2008. MR 2427530
[5] Leonard Evens, The cohomology ring of a finite group, Trans. Amer. Math. Soc. 101 (1961), 224-239. MR 137742
[6] _ The cohomology of groups, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1991, Oxford Science Publications. MR 1144017
[7] Skip Garibaldi, Alexander Merkurjev, and Jean-Pierre Serre, Cohomological invariants in Galois cohomology, University Lecture Series, vol. 28, American Mathematical Society, Providence, RI, 2003. MR 1999383
[8] Matthew Gherman and Alexander Merkurjev, Negligible degree two cohomology of finite groups, J. Algebra 611 (2022), 82-93. MR 4472425
[9] Philippe Gille and Tamás Szamuely, Central simple algebras and Galois cohomology, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006. MR 2266528
[10] Christian Haesemeyer and Charles A. Weibel, The norm residue theorem in motivic cohomology, Annals of Mathematics Studies, vol. 200, Princeton University Press, Princeton, NJ, 2019. MR 3931681
[11] Kazuya Kato, Galois cohomology of complete discrete valuation fields, Algebraic K-Theory: Proceedings of a Conference Held at Oberwolfach, June 1980 Part II, Springer, 2006, pp. 215-238.
[12] T. Y. Lam, Introduction to quadratic forms over fields, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005. MR 2104929
[13] D. Orlov, A. Vishik, and V. Voevodsky, An exact sequence for $K_{*}^{M} / 2$ with applications to quadratic forms, Ann. of Math. (2) 165 (2007), no. 1, 1-13. MR 2276765
[14] D. Quillen and B. B. Venkov, Cohomology of finite groups and elementary abelian subgroups, Topology 11 (1972), 317-318. MR 294506
[15] Daniel Quillen, The spectrum of an equivariant cohomology ring. I, II, Ann. of Math. (2) 94 (1971), 549-572; ibid. (2) 94 (1971), 573-602. MR 298694
[16] Jean-Pierre Serre, Galois cohomology, english ed., Springer Monographs in Mathematics, SpringerVerlag, Berlin, 2002, Translated from the French by Patrick Ion and revised by the author. MR 1867431
[17] , Oeuvres/Collected papers. IV. 1985-1998, Springer Collected Works in Mathematics, Springer, Heidelberg, 2013, Reprint of the 2000 edition [MR1730973]. MR 3185222

Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA
Email address: mattgherman3@gmail.com
Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA Email address: merkurev@math.ucla.edu


[^0]:    Date: April, 2023.
    Key words and phrases. Negligible cohomology, Galois cohomology, formally real fields. Mathematical Subject Classifications (2020): 12G05, $20 J 06$.

    The work of the second author has been supported by the NSF grant DMS \#1801530.

