EQUIVARIANT K-THEORY

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1. Equivariant K-theory

The equivariant K-theory was developed by R. Thomason in [12].

1.1. **Definitions.** Let G be an algebraic group over a field F. A scheme X over F is called a *G-scheme* if an action morphism $\theta : G\times X \to X$ of the group G on X is given, which satisfies the usual associative and unital identities for an action. In other words, to give a structure of a G-scheme on a scheme X is to give, for every commutative F-algebra R, a natural in R action of the group of R-point $G(R)$ on the set $X(R)$.

A *G-module M over X* is a quasi-coherent $\mathcal{O}_X$-module M together with an isomorphism of $\mathcal{O}_{G\times X}$-modules

$$\rho = \rho_M : \theta^*(M) \xrightarrow{\sim} p_2^*(M),$$

(where $p_2 : G \times X \to X$ is the projection), satisfying the cocycle condition

$$p_{23}^*(\rho) \circ (\mathrm{id}_G \times \theta)^*(\rho) = (m \times \mathrm{id}_X)^*(\rho),$$

where $p_{23} : G \times G \times X \to G \times X$ is the projection and $m : G \times G \to G$ is the multiplication (see [**?**, Ch. 1, §3] or [12]).

A morphism $\alpha : M \to N$ of G-modules is called a *G-morphism* if

$$\rho_N \circ \theta^*(\alpha) = p_2^*(\alpha) \circ \rho_M.$$

Let M be a quasi-coherent $\mathcal{O}_X$-module. For a point $x : \operatorname{Spec} R \to X$ of X over a commutative F-algebra R, write $M(x)$ for the R-module of global sections of the sheaf $x^*(M)$ over $\operatorname{Spec} R$. Thus, M defines the functor sending R to the family $\{M(x)\}$ of R-modules indexed by the R-valued point $x \in X(R)$. To give a G-module structure on M is to give natural in R isomorphisms of R-modules

$$\rho_{g,x} : M(x) \to M(gx)$$

for all $g \in G(R)$ and $x \in X(R)$ such that $\rho_{gg',x} = \rho_{g,g'x} \circ \rho_{g',x}$.

Example 1.1. Let X be a G-scheme. A *G-vector bundle on X* is a vector bundle $E \to X$ together with a linear G-action $G \times E \to E$ compatible with the one on X. The sheaf of sections P of a G-vector bundle E has a natural structure of a G-module. Conversely, a G-module structure on the sheaf P of sections of a vector bundle $E \to X$ yields a structure of a G-vector bundle on E. Indeed, for a commutative F-algebra R and a point $x \in X(R)$ the fiber of the map $E(R) \to X(R)$ over x is canonically isomorphic to $P(x)$.

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We write $\mathcal{M}(G;X)$ for the abelian category of coherent G-modules over a G-scheme X and G-morphisms. We set for every $n \geq 0$:

$$K'_n(G;X) = K_n(\mathcal{M}(G;X)).$$

A flat morphism $f : X \to Y$ of schemes over F induces the exact functor

$$\mathcal{M}(G;Y) \to \mathcal{M}(G;X), \quad M \mapsto f^*(M)$$

and therefore defines the *pullback* homomorphism

$$f^* : K'_n(G;Y) \to K'_n(G;X).$$

A G-projective morphism $f : X \to Y$ is a morphism that factors equivariantly as a closed embedding into the projective bundle space $\mathbb{P}(E)$, where E is a G-vector bundle on Y. Such a morphism f yields the *push-forward* homomorphism

$$f_* : K'_n(G;X) \to K'_n(G;Y)$$

[12, 1.5].

If G is the trivial group, then $\mathcal{M}(G;X) = \mathcal{M}(X)$ is the category of coherent sheafs on X and therefore, $K'_n(G;X) = K'_n(X)$.

Consider the full subcategory $\mathcal{P}(G;X)$ in $\mathcal{M}(G;X)$ consisting of locally free $\mathcal{O}_X$-modules. This category is naturally equivalent to the category of vector G-vector bundles over X (see Example 1.1) The category $\mathcal{P}(G;X)$ has a natural structure of an exact category. We set

$$K_n(G;X) = K_n(\mathcal{P}(G;X)).$$

The functor $K_n(G;*)$ is contravariant with respect to arbitrary G-morphisms of G-varieties. If G is a trivial group, we have $K_n(G;X) = K_n(X)$.

The inclusion of categories $\mathcal{P}(G;X) \hookrightarrow \mathcal{M}(G;X)$ induces a homomorphism

$$K_n(G;X) \to K'_n(G;X).$$

Example 1.2. Let $\mu : G \to \mathbf{GL}(V)$ be a finite dimensional representation of an algebraic group G over a field F. One can view the G-module V as a G-vector bundle over $\operatorname{Spec} F$. Clearly, we obtain an equivalence of the category $\operatorname{Rep}(G)$ of finite dimensional representations of G and the categories $\mathcal{P}(G;\operatorname{Spec} F) = \mathcal{M}(G;\operatorname{Spec} F)$. Hence there are natural isomorphisms

$$R(G) \xrightarrow{\sim} K_0(G;\operatorname{Spec} F) \xrightarrow{\sim} K'_0(G;\operatorname{Spec} F),$$

where $R(G) = K_0\big(\operatorname{Rep}(G)\big)$ is the *representation ring* of G. Therefore, for every G-scheme X over F the pullback map $R(G) \to K_0(G;X)$ with respect to the structure morphism $X \to \operatorname{Spec} F$ is a ring homomorphism, making $K_0(G;X)$ (and similarly $K'_0(G;X)$) a module over $R(G)$.

Let H be a (closed) subgroup of an algebraic group G over F. By restriction of the structure, we obtain exact functors

$$\mathcal{M}(G;X) \to \mathcal{M}(H;X), \qquad \mathcal{P}(G;X) \to \mathcal{P}(H;X)$$

inducing group homomorphisms

$$K'_n(G;X) \to K'_n(H;X), \qquad K_n(G;X) \to K_n(H;X).$$

We call these maps the *restriction homomorphisms.*

1.2. **Torsors.** Let G be an algebraic group over F and let $p : X \to Y$ be a G-torsor. Then for any coherent $\mathcal{O}_Y$-module M there is a natural G-module structure on $p^*(M)$ over X given by the identity automorphism of $\theta^* p*(M) = p_2^* p^*(M)$. Thus, there is an exact functor check!

$$p^0 : \mathcal{M}(Y) \to \mathcal{M}(G; X), \qquad M \mapsto p^*(M).$$

The following statement is well-known (see, for example [6, Prop. 3.3]).

Proposition 1.3. *The functor p^0 is an equivalence of categories. In particular, the homomorphism $p^* : K'_n(Y) \to K'_n(G; X)$, induced by p^0, is an isomorphism.*

Proof. Under the isomorphisms

$$G \times X \tilde{\to} X \times_Y X, \qquad (g, x) \mapsto (gx, x),$$

$$G \times G \times X \tilde{\to} X \times_Y X \times_Y X, \qquad (g, g', x) \mapsto (gg'x, g'x, x)$$

the action morphism θ is identified with the first projection $p_1 : X \times_Y X \to X$ and the morphisms $m \times \mathrm{id}$, $\mathrm{id} \times \theta$ are identified with the projections $p_{13}, p_{12} : X \times_Y X \times_Y X \to X \times_Y X$. Hence, the isomorphism ρ giving an G-module structure on a O_X-module M can be identified with the *descent data*, i.e. with an isomorphism

$$\varphi : p_1^*(M) \tilde{\to} p_2^*(M)$$

of $O_{X \times_Y X}$-modules satisfying the usual cocycle condition

$$p_{23}^*(\varphi) \circ p_{12}^*(\varphi) = p_{13}^*(\varphi) \ .$$

The statement follows now from the theory of faithfully flat descent [7, Prop.2.22]. □

Proposition 1.2 allows us to consider K-theory of homogeneous varieties as equivariant K-theory of algebraic groups:

Corollary 1.4. *Let H be a closed subgroup of an algeraic group G. Then G is an H-scheme by left translations and there is a natural isomorphism $p^* : K'_n(G/H) \tilde{\to} K'_n(H; G)$.*

1.3. **Basic results in equivariant K-theory.** We formulate basic statements in the equivariant algebraic K-theory developed by R. Thomason in [12]. In all of them G is an algebraic group over a field F and X is a G-scheme.

Let $Z \subset X$ be a closed G-subscheme and let $U = X - Z$. Since every coherent G-module on U extends to a coherent G-module on X [12, Cor. 2.4], the category $\mathcal{M}(G; U)$ is equivalent to the factor category of $\mathcal{M}(G; X)$ by the subcategory $\mathcal{M}'$ of coherent G-modules supported on Z. By Quillen's devissage theorem [9, §5], the inclusion of categories $\mathcal{M}(G; Z) \subset \mathcal{M}'$ induces an isomorphism $K'_n(\mathcal{M}') \xrightarrow{\sim} K'_n(G;Z)$. The localization in algebraic K-theory [9, ???] yields the connecting homomorphisms

$$K'_{n+1}(G;U) \xrightarrow{\delta} K'_n(\mathcal{M}') \xrightarrow{\sim} K'_n(G;Z)$$

and by [9, ???] we get

Theorem 1.5. [12, Th. 2.7] (Localization) *The sequence*

$$\to K'_{n+1}(G;U) \xrightarrow{\delta} K'_n(G;Z) \xrightarrow{i_*} K'_n(G;X) \xrightarrow{j^*} K'_n(G;U) \xrightarrow{\delta} \dots,$$

where $i : Z \to X$ and $j : U \to X$ are imbeddings, is exact.

Corollary 1.6. *Let X be a G-scheme. Then the natural closed G-embedding $f : X_{red} \to X$ induces isomorphism $f_* : K_n(G;X_{red}) \to K_n(G;X)$.*

Let X be a G-scheme and let E be a G-vector bundle of rank $r+1$ over X. The projective bundle $\mathbb{P}(E)$ has a natural structure of a G-scheme so that the natural morphism $p : \mathbb{P}(E) \to X$ is G-equivariant. We write $\mathcal{O}(-1)$ for the G-module of sections of the tautological line bundle on $\mathbb{P}(E)$ and set $\mathcal{O}(-i) = \mathcal{O}(-1)^{\otimes i}$.

A modification of the Quillen's proof [9, §8] of the standard projective bundle theorem yields:

Theorem 1.7. [12, Th. 3.1] (Projective Bundle Theorem) *The correspondence*

$$(a_0, a_1, \dots, a_r) \mapsto \sum_{i=0}^{r} [\mathcal{O}(-i)] \otimes p^* a_i$$

induces isomorphisms

$$\coprod_{i=0}^{r} K_n(G;X) \to K_n\big(G;\mathbb{P}(E)\big), \quad \coprod_{i=0}^{r} K'_n(G;X) \to K'_n\big(G;\mathbb{P}(E)\big).$$

Let X be a G-scheme and let E be a G-vector bundle over X. Let $f : Y \to X$ be a torsor under the vector bundle space E (considered as a group scheme over X) and G acts on Y so that f and the action map $E \times_X Y \to Y$ are G-equivariant. For example, one can take the trivial torsor $Y = E$.

Theorem 1.8. [12, Th. 4.1] (Strong Homotopy Invariance Property) *The pullback homomorphism*

$$f^* : K'_n(G;X) \to K'_n(G;Y)$$

is an isomorphism.

The idea of the proof is construct an exact sequence of G-vector bundles

$$0 \to E \to V \to \mathbf{1} \xrightarrow{\varphi} 0$$

such that $\varphi^{-1}(1) \simeq Y$. Thus, Y is isomorphic to the open complement of the projective bundle $\mathbb{P}(E)$ in $\mathbb{P}(V)$. Then one uses the Projective bundle Theorem and the localization to compute equivariant K-theory of Y.

Corollary 1.9. *If G acts linearly on the affine space $\mathbb{A}^n_F$, then the projection $p : X \times \mathbb{A}^n_F \to X$ induces the inverse image isomorphism*

$$p^* : K'_n(G; X) \xrightarrow{\sim} K'_n(G; X \times \mathbb{A}^n_F) .$$

Theorem 1.10. [12, Th. 5.7] (Duality for smooth schemes) *Let X be a smooth G-scheme over F. Then the canonical homomorphism $K_n(G; X) \to K'_n(G; X)$ is an isomorphism.*

Theorem 1.11. [12, Prop. 6.2] (Faddeev-Shapiro Lemma) *Let H be a closed subgroup in G and let $Y = G/H$. Then the restriction along the H-map $X \simeq X \times \{H\} \hookrightarrow X \times Y$ induces equivalences of categories*

$$\mathcal{P}(G; X \times Y) \xrightarrow{\sim} \mathcal{P}(H; X), \qquad \mathcal{M}(G; X \times Y) \xrightarrow{\sim} \mathcal{M}(H; X).$$

In particular, there are natural isomorphisms

$$K_n(G; X \times Y) \xrightarrow{\sim} K_n(H; X), \qquad K'_n(G; X \times Y) \xrightarrow{\sim} K'_n(H; X).$$

For any representation $\rho : H \to \mathbf{GL}(V)$ one can associate the G-vector bundle $P_\rho = (G \times V)/H$ over $Y = G/H$, where the H-action on $G \times V$ is given by $h \cdot (g, v) = (gh^{-1}, \rho(h)(v))$.

Corollary 1.12. *The assignment $\rho \mapsto [P_\rho]$ induces an isomorphism*

$$R(H) \xrightarrow{\sim} K_0(G; Y) \simeq K'_0(G; Y).$$

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2. Category $\mathcal{C}(G)$ of G-equivariant K-correspondences

Let G be an algebraic group over a field F and let A be a separable F-algebra. A *G-A-module* over a G-scheme X is a G-module M over X being also a left $A \otimes_F \mathcal{O}_X$-module such that the G action on M is A-linear.

We consider the abelian category $\mathcal{M}(G; X, A)$ of G-A-modules and morphisms of $A \otimes_F O_X$- and G- modules. Set

$$K'_n(G; X, A) = K_n(\mathcal{M}(G; X, A)) .$$

The functor $K'_n(G; ?, A)$ is contravariant with respect to flat G-morphisms and is covariant with respect to projective G-morphisms of G-schemes. The group $K'_n(G; X, F)$ coincides with $K'_n(G; X)$.

Consider also the full subcategory $\mathcal{P}(G; X, A)$ in $\mathcal{M}(G; X, A)$ consisting of G-A-modules which are locally free $\mathcal{O}_X$-modules. The K-groups of the category $\mathcal{P}(G; X, A)$ we denote by

$$K_n(G; X, A) = K_n(\mathcal{P}(G; X, A)) .$$

The group $K_n(G; X, F)$ coincides with $K_n(G; X)$.

In [8] I. Panin has defined the *category of G-equivariant K-correspondences* $\mathcal{C}(G)$ whose objects are the pairs (X, A), where X is a smooth projective G-scheme over F and A is a separable F-algebra. Morphisms in $\mathcal{C}(G)$ are defined as follows:

$$\operatorname{Hom}_{\mathcal{C}(G)}((X, A), (Y, B)) = K_0(G; X \times Y, A^{op} \otimes_F B).$$

If $u : (X, A) \to (Y, B)$ and $v : (Y, B) \to (Z, C)$ are two morphisms in $\mathcal{C}(G)$, then their composition is defined by the formula

$$v \circ u \overset{\text{def}}{=} p_{13*}(p_{23}^*(v) \otimes_B p_{12}^*(u)),$$

where p_{12}, p_{13} and p_{23} are projections from $X \times Y \times Z$ to $X \times Y$, $X \times Z$ and $Y \times Z$ respectively. The identity endomorphism of (X, A) in $\mathcal{C}(G)$ is the class $[A \otimes_F \mathcal{O}_\Delta]$, where $\Delta \subset X \times X$ is the diagonal, in the group

$$K_0'(G; X \times X, A^{op} \otimes_F A) = K_0(G; X \times X, A^{op} \otimes_F A) = \operatorname{End}_{\mathcal{C}(G)}(X, A).$$

We will simply write X for (X, F) and A for $(\operatorname{Spec} F, A)$ in $\mathcal{C}(G)$.

The category $\mathcal{C}(G)$ for the trivial group G is simply denoted by $\mathcal{C}$. There is the forgetful functor $\mathcal{C}(G) \to \mathcal{C}$.

For every scheme Z over F and every $n \in \mathbb{Z}$ there is the *realization functor*

$$\mathcal{K}_n^Z : \mathcal{C}(G) \to \textbf{Abelian Groups},$$

taking a pair (X, A) to $K_n'(G; X \times Z, A)$ and a morphism

$$v \in \operatorname{Hom}_{\mathcal{C}(G)}((X, A), (Y, B)) = K_0(G; X \times Y, A^{op} \otimes_F B)$$

to

$$\mathcal{K}_n^Z(v) : K_n'(G; X \times Z, A) \to K_n'(G; Y \times Z, B)$$

given by the formula

$$\mathcal{K}_n^Z(v)(u) = v \circ u.$$

We simply write $\mathcal{K}_n$ for $\mathcal{K}_n^{\operatorname{Spec} F}$.

Example 2.1. Let X be a smooth projective scheme over F. The identity in $K_0(X)$ defines two morphisms $u : X \to \operatorname{Spec} F$ and $v : \operatorname{Spec} F \to X$ in $\mathcal{C}$. If $p_*[\mathcal{O}_X] = 1 \in K_0(F)$, where $p : X \to \operatorname{Spec} F$ is the structure morphism (for example, if X is a projective homogeneous variety), then the composition $u \circ v$ in $\mathcal{C}$ is the identity. In other words, the morphism p splits canonically in $\mathcal{C}$, i.e., the point $\operatorname{Spec} F$ is a canonical "direct summand" of X in $\mathcal{C}$, although X may have no rational points. In particular, for any scheme Z over F, the group $K_n(Z)$ is a canonical direct summand of $K_n(X \times Z)$.

Let G be a split reductive group over a field F with simply connected commutator subgroup and let $B \subset G$ be a Borel subgroup. By [11, Th.1.3], $R(B)$ is a free $R(G)$-module.

The following statement is a slight generalization of [8, Th. 6.6].

Proposition 2.2. *Let G be a split reductive group over a field F with simply connected commutator subgroup, let $B \subset G$ be a Borel subgroup and set $Y = G/B$. Choose a basis $u_1, u_2, \ldots, u_m$ of $R(B) = K_0(G; Y)$ over $R(G)$. Then the element*

$$u = (u_i) \in R(B)^m = K_0(G; Y, F^m)$$

defines an isomorphism $F^m \xrightarrow{\sim} Y$ in the category $\mathcal{C}(G)$.

Proof. Denote by $p : G/B \to \operatorname{Spec} F$ the structure morphism. Since G/B is a projective variety, the direct image

$$p_* : R(B) = K_0'(G; G/B) \to K_0'(G; \operatorname{Spec} F) = R(G)$$

is well defined. The $R(G)$-bilinear form on $R(B)$ defined y the formula

$$\langle u, v\rangle_G = p_*(u \cdot v)$$

is unimodular ([3], [8, Th. 8.1.], [5, Prop. 2.17]).

Let $v_1, v_2, \ldots, v_m$ be the dual $R(G)$-base of $R(B)$ with respect to the unimodular bilinear form $\langle\ ,\ \rangle_G$. The element $v = (v_i) \in K_0(G; Y, F^m)$ can be considered as a morphism $Y \to F^m$ in $\mathcal{C}(G)$. The fact that u and v are dual bases is equivalent to $v \circ u = \mathrm{id}$. In order to prove that $u \circ v = \mathrm{id}$ it suffices to show that the $R(G)$-module $K_0(G, Y \times Y)$ is generated by m^2 elements (see [8, Cor. 7.3]). It is proved in [8, Prop. 8.4] for a simply connected group G, but the proof goes through for a reductive group G with simply connected commutator subgroup.

3. K-theory of projective homogeneous varieties

Theorem 3.1. *Let G be a simply connected group over a field F and let X be a projective homogeneous G-variety. There exist a separable F-algebra A and an isomorphism $X \simeq A$ in the category $\mathcal{C}(G)$. In particular, $K_*(G; X) \simeq K_*(G; A)$ and $K_*(X) \simeq K_*(A)$.*

Corollary 3.2. *The restriction homomorphism $K_0(G; X) \to K_0(X)$ is surjective.*

Proof. Follows from the surjectivity of the restriction homomorphism $K_0(G; A) \to K_0(A)$. □

We will generalize the corollary in ???.

4. K-theory of toric varieties

Assume that a torus T acts on a normal geometrically irreducible variety X defined over a field F. The variety X is called a *toric T-variety* if there is an open orbit which is a principal homogeneous space over T. A toric T-variety is called a *toric model of T* or *toric T-model* if the open orbit has a rational point. A choice of a rational point x in the open orbit gives an open T-equivariant imbedding $T \hookrightarrow X$, $t \mapsto tx$.

4.1. **K-theory of toric models.**

Proposition 4.1. [6, Prop. 5.6] *Let X be a smooth toric T-model defined over a field F. Then there is a torus S over F, an S-torsor $\pi : U \to X$ and an S-equivariant open imbedding of U into an affine space $\mathbb{A}$ on which S acts linearly.*

Remark 4.2. It turns out that the canonical homomorphism $S^*_{\mathrm{sep}} \to \mathrm{Pic}(X_{\mathrm{sep}})$ is an isomorphism, so that $\pi : U \to X$ is the *universal torsor* in the sense of [2, 2.4.4]. Thus, the Proposition 4.1 asserts that the universal torsor of X is equivariantly imbedded into an affine space as an open set.

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Let $\rho : S \to GL(W)$ be a representation over F. Assume that there is an action of an étale F-algebra A on W commuting with the action of S. Then A acts on the vector bundle P_ρ, therefore, P_ρ defines an element $u_\rho \in K_0(X, A)$, i.e., a morphism $u_\rho : A \to X$ in $\mathcal{C}$. The composition

$$K_0(A) \xrightarrow{\alpha_\rho} R(S) \xrightarrow{r} K_0(X)$$

where α_ρ is induced by the exact functor $M \mapsto M \otimes_A W$, is clearly given by the rule $x \mapsto u_\rho \circ x$.

Let $\rho : S \to GL(W)$ be an irreducible representation over F. Since S is a torus, ρ is a corestriction from a finite separable field extension L_ρ/F of a 1-dimensional representation of S. Thus, there is an action of L_ρ on W commuting with the action of S. Note that the element u_ρ defined above is represented by an element of the Picard group $\mathrm{Pic}(X \otimes_F L_\rho)$.

Now we consider two arbitrary irreducible representations ρ and μ of the torus S over F, and apply the construction described above to the torus $S \times S$ and its representation

$$\rho \otimes \mu : S \times S \to GL(W_\rho \otimes_F W_\mu) \ .$$

The composition

$$K_0(L_\rho \otimes_F L_\mu) \xrightarrow{\alpha_{\rho,\mu}} R(S \times S) \xrightarrow{r} K_0(X \times X)$$

coincides with the map

$$x \mapsto u_\rho^{op} \circ x \circ u_\mu \ ,$$

where the composition is taken in $\mathcal{C}$ and $u_\mu : X \to L_\mu$, $u_\rho^{op} : L_\rho \to X$, $x : L_\mu \to L_\rho$ are considered as the morphisms in $\mathcal{C}$.

Now let Φ be a finite set of irreducible representations of S. Set

$$A = \prod_{\rho \in \Phi} L_\rho, \quad u = \sum_{\rho \in \Phi} u_\rho, \quad \alpha = \sum_{\rho,\mu \in \Phi} \alpha_{\rho,\mu} \ .$$

The element u_ρ is represented by an element of the Picard group $\mathrm{Pic}(X \otimes_F A)$.

Then the composition

$$K_0(A \otimes_F A) \xrightarrow{\alpha} R(S \times S) \xrightarrow{r} K_0(X \times X)$$

is given by the rule $x \mapsto u^{op} \circ x \circ u$ where u is considered as a morphism $X \to L$.

The homomorphism r coincides with the composition

$$R(S\times S)=K_0(S\times S;\operatorname{Spec}F)\xrightarrow{\sim}K_0(S\times S;\mathbb{A}\times\mathbb{A})\twoheadrightarrow\\ K_0(S\times S;U\times U)=K_0(X\times X)$$

and hence r is surjective. By the representation theory of algebraic tori, the sum of all the $\alpha_{\rho,\mu}$ is an isomorphism. It follows that for sufficiently large (but finite!) set Φ of irreducible representations of S the identity $\mathrm{id}_X\in K_0(X\times X)$ lies in the image of $r\circ\alpha$. In other words, there exists $x\in K_0(A\otimes_F A)$ such that $u^{op}\circ x\circ u=\mathrm{id}_X$ in $\mathcal{C}$, i.e. $v=u^{op}\circ x$ is a left inverse to $u:X\to A$ in $\mathcal{C}$. We have proved the following

Theorem 4.3. [6] *Let X be a smooth projective toric model of an algebraic torus defined over a field F. Then there exist an étale F-algebra A and elements u, $v\in K_0(X,A)$ such that the composition $X\xrightarrow{u}A\xrightarrow{v}X$ in $\mathcal{C}$ is the identity and u is represented by a class in* $\mathrm{Pic}(X\otimes_F A)$.

4.2. **K-theory of toric varieties.** Let T be a torus over F. The natural G-equivariant bilinear map

$$T(F_{\mathrm{sep}})\otimes T^*_{\mathrm{sep}}\to F^{\times}_{\mathrm{sep}},\qquad x\otimes\chi\mapsto\chi(x)$$

induces the pairing of Galois cohomology groups

$$H^1\big(F,T(F_{\mathrm{sep}})\big)\otimes H^1(F,T^*_{\mathrm{sep}})\longrightarrow H^2(F,F^{\times}_{\mathrm{sep}})=\mathrm{Br}(F),$$

where $Br(F)$ is the Brauer group of F. There is a natural isomorphism $Pic(T)\simeq H^1\big(F,T(F_{\mathrm{sep}})\big)$. A principal homogeneous T-space U defines an element $[U]\in H^1\big(F,T(F_{\mathrm{sep}})\big)$. Therefore, the pairing induces the homomorphism

$$\lambda^U:\mathrm{Pic}(T)\to\mathrm{Br}(F),\qquad [Q]\mapsto[U]\cup[Q].$$

Let X be a toric variety of torus T. Let U be the open orbit which is a principal homogeneous space over T.

Theorem 4.4. [6, Th. 7.6] *Let Y be a smooth projective toric variety over a field F. Then there exist an étale F-algebra A, a separable F-algebra B of rank n^2 over its center A and morphisms $u:Y\to B$, $v:B\to Y$ in $\mathcal{C}$ such that $v\circ u=\mathrm{id}$. The morphism u is represented by a locally free sheaf in $\mathcal{P}(Y,B)$ of rank n. The class of the algebra B in $\mathrm{Br}(A)$ belongs to the image of $\lambda^{U_A}:\mathrm{Pic}(T_A)\to\mathrm{Br}(A)$.*

Theorem says that Y is a “direct summand” of B.

5. Equivariant K-theory of solvable algeraic groups

We consider separately equivariant K-theory of unipotent groups and algeraic tori.

5.1. **Split unipotent groups.** A unipotent group U is called *split* if there is a chain of subgroups of U with subsequent factor groups isomorphic to the additive group $\mathbf{G}_a$. For example, the unipotent radical of a Borel subgroup of a (quasisplit) reductive group is split.

Theorem 5.1. *Let U be a split unipotent group and let X be a U-scheme. Then the restriction homomorphism $K'_n(U;X) \to K'_n(X)$ is an isomorphism.*

Proof. Since U is split, it is sufficient to prove that for a subgroup $U' \subset U$ with $U/U' \simeq \mathbf{G}_a$ the restriction homomorphism $K'_n(U;X) \to K'_n(U';X)$ is an isomorphism. By Theorem 1.11, this homomorphism coincides with the pullback $K'_n(U;X) \to K'_n(U;X\times\mathbf{G}_a)$ with respect to the projection $X\times\mathbf{G}_a \to X$ that is an isomorphism by the Homotopy Invariance property Theorem 1.8. □

5.2. **Split algeraic tori.** Let T be a split torus over a field F. Choose a basis $\chi_1, \chi_2, \dots, \chi_r$ of the character group $T*$. We define an action of T on the affine space $\mathbb{A}^r_F$ by the rule $b\cdot x = y$ where $y_i = \chi_i(b)x_i$. Denote by V_i $(i = 1, 2, \dots, r)$ the coordinate hyperplane in $\mathbb{A}^r_F$ defined by the equation $x_i = 0$. Clearly, V_i is a closed B-subvariety in $\mathbb{A}^r_F$ and $T = \mathbb{A}^r_F - \cup_{i=1}^r V_i$.

In [4] M. Levine has constructed a spectral sequence associated to a family of closed subschemes in a given scheme. This sequence generalizes the localization exact sequence. We adapt this sequence to the equivariant algebraic K-theory and also change indices making this sequence of homological type.

Let X be a G-scheme over F. The family of closed subsets $Z_i = X \times V_i$ in $X \times \mathbb{A}^r_F$ induces then the following spectral sequence

$$E^1_{p,q} = \coprod_{|I|=p} K'_q(T; X\times V_I) \Rightarrow K'_{p+q}(T; X\times T),$$

where $V_I = \cap_{i\in I} V_i$.

By Theorem 1.11, the group $K'_{p+q}(T; X\times T)$ is isomorphic to $K'_{p+q}(X)$.

In order to compute $E^1_{p,q}$, note that V_I is an affine space over F, hence the inverse image $K'_q(T;X) \to K'_q(T; X\times V_I)$ is an isomorphism by the himitiry Invariance property Theorem 1.8. Thus,

$$E^1_{p,q} = \coprod_{|I|=p} K'_q(T;X)\cdot e_I$$

and by [4, p.419] the differential map $d : E^1_{p+1,q} \to E^1_{p,q}$ is given by the formula

$$d(x\cdot e_I) = \sum_{k=0}^{p} (-1)^k (1-\chi_k^{-1}) x\cdot e_{I-\{i_k\}}, \tag{1}$$

where $I = \{i_0 < i_1 < \dots < i_p\}$.

Consider Kozsul complex C_* built of the free module $R(T)^r$ over $R(T)$ and the system of elements $1-\chi_i^{-1} \in R(T)$. More precisely,

$$C_p = \coprod_{|I|=p} R(T)\cdot e_I$$

and the differential $d : C_{p+1} \to C_p$ given by the rule formally equal to (1) with $x \in R(T)$.

The representation ring $R(T)$ is the group ring over the character group $R(T)$. The Kozsul complex gives a resolution $C_* \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ by free $R(T)$-modules. It follows from (1) that the complex $E^1_{*,q}$ coincides with

$$C_* \otimes_{R(T)} K'_q(T; X).$$

Hence, the term $E^2_{p,q}$, being the homology group of $E^1_{*,q}$, equals

$$\mathrm{Tor}_p^{R(T)}(\mathbb{Z}, K'_q(T; X)).$$

Theorem 5.2. *Let T be a split torus over a field F and let X be a G-scheme. Then there exists a spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}_p^{R(T)}(\mathbb{Z}, K'_q(T; X)) \Rightarrow K'_{p+q}(X).$$

We are going to prove that if X is smooth projective, the spectral sequence splits.

Let G be an algebraic group and let $H \subset G$ be a closed subgroup with a group homomorphism $\pi : G \to H$ that is the identity on H. For a smooth projective G-scheme X we write $\dot{X}$ for the scheme X together with the G-action $(g, x) \mapsto= \pi(g)x$.

Lemma 5.3. *If the restriction homomorphism $K_0(G; \dot{X} \times X) \to K_0(H; X \times X)$ is surjective, then the restriction homomorphism $K'_n(G; X) \to K'_n(H; X)$ is a split surjection.*

Proof. Since the restriction map

$$\mathrm{res}_{G/H} : \mathrm{Hom}_{\mathcal{C}(G)}(\dot{X}, X) = K_0(G; \dot{X} \times X) \to$$
$$K_0(H; X \times X) = \mathrm{Hom}_{\mathcal{C}(H)}(X, X)$$

is surjective, there is $v \in \mathrm{Hom}_{\mathcal{C}(G)}(\dot{X}, X)$ such that $\mathrm{res}_{G/H}(v) = \mathrm{id}_X$.

Consider the diagram

$$\begin{array}{ccccc} K'_n(H; X) & \xrightarrow{\pi^*} & K'_n(G; \dot{X}) & \xrightarrow{\mathcal{K}_n(v)} & K'_n(G; X) \\ & & {\scriptstyle \mathrm{res}}\downarrow & & {\scriptstyle \mathrm{res}}\downarrow \\ & & K'_n(H; X) & = & K'_n(H; X), \end{array}$$

where the square is commutative since $\mathrm{res}_{G/H}(v) = \mathrm{id}_X$. The equality $\mathrm{res} \circ \pi^* = \mathrm{id}$ implies that the composition in the top row splits the restriction homomorphism.

Let T be a split torus over F, $\chi \in T^*$ be a character such that $T^*/(\mathbb{Z} \cdot \chi)$ is a torsion-free group. Then $T' = \ker(\chi)$ is a subtorus in T. Denote by $\pi : T \to T'$ any splitting of the imbedding $T' \hookrightarrow T$.

Proposition 5.4. *Let X be a smooth projective T-scheme. Then the restriction homomorphism $K'_n(T; X) \to K'_n(T'; X)$ is a split surjection.*

Proof. We use the notation $\dot{X}$ as above. Since $T/T' \simeq \mathbb{G}_m$, by Corollary 1.9, Theorem 1.11 and the localization (Theorem 1.5), we have the following surjection

$$K_0'(T;\dot{X}\times X) \xrightarrow{\sim} K_0'(T;\dot{X}\times X\times\mathbb{A}_F^1) \twoheadrightarrow K_0'(T;\dot{X}\times X\times\mathbb{G}_m) \simeq K_0'(T';\dot{X}\times X)$$

which is nothing but the restriction homomorphism. The statement follows from Lemma 5.3.

Corollary 5.5. *The sequence*

$$0 \to K_n'(T;X) \xrightarrow{1-\chi} K_n'(T;X) \xrightarrow{\text{res}} K_n'(T';X) \to 0$$

is split exact.

Proof. We consider $X \times \mathbb{A}_F^1$ as a G-scheme with respect to the action of G on $\mathbb{A}_F^1$ given by the multiplication by χ. In the localization exact sequence

$$\ldots \to K_n'(G;X) \xrightarrow{i_*} K_n'(G;X\times\mathbb{A}_F^1) \xrightarrow{j^*} K_n'(G;X\times\mathbb{G}_m) \xrightarrow{\delta} \ldots$$

where $i : X = X\times\{0\} \hookrightarrow X\times\mathbb{A}_F^1$ and $j : X\times\mathbb{G}_m \hookrightarrow X\times\mathbb{A}_F^1$ are imbeddings, the second term is identified with $K_n'(G;X)$ by Corollary 1.9 and the third one with $K_n'(H;X)$ since $G/H \simeq \mathbb{G}_m$ as G-schemes (see Theorem 1.11). With this identifications j^* is the restriction homomorphism which is a split surjection by Proposition 5.4. In order to determine i_*, by projection formula, i_* is the multiplication by $i_*(1)$. Let t be the coordinate of $\mathbb{A}^1$. It follows from the exactness of the sequence of G-modules over $X \times \mathbb{A}_F^1$

$$0 \to \mathcal{O}_{X\times\mathbb{A}^1}[\chi^{-1}] \xrightarrow{t} \mathcal{O}_{X\times\mathbb{A}^1} \to i_*(\mathcal{O}_X) \to 0$$

that $i_*(1) = 1-\chi^{-1}$. □

Theorem 5.6. *Let T be a split torus and let X be a smooth projective scheme. Then the spectral sequence in Theorem 5.7 degenerates. i.e.,*

$$\operatorname{Tor}_p^{R(T)}(\mathbb{Z}, K_n'(T;X)) = \begin{cases} K_n'(X), & \text{if } p=0; \\ 0, & \text{if } p>0. \end{cases}$$

check

Proof. Let $\chi_1, \chi_2 \ldots, \chi_r$ be a $\mathbb{Z}$-base of the character group T^*. Since $R(B)$ is a Laurent polynomial ring in variables χ_i, the elements $1-\chi_i \in R(B)$ form a $R(B)$-regular sequence, the result follows from [**?**, IV-7]. □

Let R be a commutative algebra. An R-algebra A is called *almost exterior* if there is a finite subset $\{a_1, a_2, \ldots, a_n\} \subset A$ such that the products $a_{i_1}a_{i_2}\ldots a_{i_k}$ for all $k \le n$ and all $i_1 < i_2 < \cdots < i_k$ form a basis of the R-module A.

graded commutative?

Theorem 5.7. *Let T be a split torus over a field F. Then the $K_*(F)$-algebra $K_*(T)$ is almost exterior.*

5.3. **Quasitrivial algebraic tori.** An algeraic torus T over a field F is called *quasitrivial* if the character Galois module T^* is permutation. In other words, T is isomorphic to the torus $\mathbf{GL}_1(C)$ of invertible elements of an étale F-algebra C. The torus $T = \mathbf{GL}_1(C)$ is embedded as an open subscheme of the affine space $\mathbb{A}(C)$. By the classical Homotopy Invariance and Localization, the pullback homomorphism

$$\mathbb{Z} = K_0\big(\mathbb{A}(C)\big) \to K_0(T)$$

is surjective. We have proved

Proposition 5.8. *For a quasitrivial torus T, one has $K_0(T) = \mathbb{Z} \cdot 1$.*

We generalize this statement in Theorem 5.9.

5.4. **Coflasque algebraic tori.** An algebraic torus T over F is called *coflasque* if for every field extension L/F the Galois cohomology group $H^1(L, T^*)$ is trivial (here T^* is the group of characters of T defined over L_{sep}). For example, a quasi-trivial torus is coflasque.

Recall that the Picard group of a coflasque torus is trivial.

Theorem 5.9. *Let T be a coflasque torus and let U be a principal homogeneous space of T. Then $K_0(U) = \mathbb{Z} \cdot 1$.*

Proof. Let X be a smooth projective toric model of T (it exists by [1]). The variety $Y = (X \times U)/T$ is then a toric variety of T that has an open orbit isomorphic to U.

By Theorem 4.4, there exist an étale F-algebra A, a separable F-algebra B of rank n^2 over its center A and morphisms $u : Y \to B$, $v : B \to Y$ in $\mathcal{C}$ such that $v \circ u = \mathrm{id}$. The morphism u is represented by a locally free sheaf in $\mathcal{P}(Y, B)$ of rank n. The class of the algebra B in $\mathrm{Br}(A)$ belongs to the image of $\lambda^{U_A} : \mathrm{Pic}(T_A) \to \mathrm{Br}(A)$.

Applying the realization functor we get a (split) surjection

$$\mathcal{K}_0(u^{op}) : K_0(B^{op}) \to K_0(Y).$$

The torus T is factorial, hence the group $\mathrm{Pic}(A)$ is trivial and therefore, the algebra B splits, so that $K_0(B^{op})$ is isomorphic canonically to $K_0(A)$. Under this identification we get a (split) surjection

$$\mathcal{K}_0(w^{op}) : K_0(A) \to K_0(Y),$$

where w is a certain element in $K_0(Y, A)$ represented by a locally free sheaf of rank one, i.e., by an element of $\mathrm{Pic}(Y \otimes_F A)$.

It follows that $K_0(Y)$ is generated by the direct images of classes of sheaves from $\mathrm{Pic}(Y_E)$ for all finite separable field extensions E/F. Since the inverse image homomorphism $K_0(Y) \to K_0(U)$ is surjective, the analogous statement holds for the open subset $U \subset Y$. But by [10, Prop. 6.10] there is an injection $\mathrm{Pic}(U_E) \hookrightarrow \mathrm{Pic}(T_E) = 0$, hence $\mathrm{Pic}(U_E) = 0$ and therefore $K_0(U) = \mathbb{Z} \cdot 1$.

6. Equivariant K-theory of simply connected groups

6.1. Simply connected groups of inner type.

Theorem 6.1. *Inner simply connected*

6.2. Simply connected group.

Proposition 6.2. *Let G be an algebraic group over F and let $f : X \to Y$ be a G-torsor over F. For every point $y \in Y$ let X_y be the fiber $f^{-1}(y)$ of f over y (so that X_y is a principal homogeneous space of G over the residue field $F(y)$). Assume that $K_0(X_y) = \mathbb{Z} \cdot 1$ for every point $y \in Y$. Then the restriction homomorphism $K_0'(G; X) \to K_0'(X)$ is surjective.*

Proof. We prove that the restriction homomorphism $\mathrm{res}_X : K_0'(G; X) \to K_0'(X)$ is surjective by induction on the dimension of X.

Assume that we have proved the statement for all schemes of dimension less than the dimension of X. We would like to prove that res_X is surjective.

We prove this statement by induction on the number of irreducible components of Y. Assume first that Y is irreducible. By Corollary 1.6, we may assume that Y is reduced.

Let $y \in Y$ be the generic point and $v \in K_0'(X)$. Since $K_0(X_y) = \mathbb{Z} \cdot 1$, the restriction homomorphism $K_0'(G; X_y) \to K_0'(X_y)$ is surjective. It follows that there exists a non-empty open subset $U' \subset Y$ such that the image of v in $K_0'(U)$, where $U = f^{-1}U'$, belongs to the image of the restriction homomorphism $K_0'(G; U) \to K_0'(U)$. Set $Z = X - U$ (considered as a reduced scheme). Since $\dim(Z) < \dim(X)$, the left vertical homomorphism in the commutative diagram with the exact rows

$$\begin{array}{ccccccc}
K_0'(G;Z) & \xrightarrow{i_*} & K_0'(G;X) & \xrightarrow{j^*} & K_0'(G;U) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{res}_Z} & & \downarrow{\scriptstyle \mathrm{res}_X} & & \downarrow{\scriptstyle \mathrm{res}_U} & & \\
K_0'(Z) & \xrightarrow{i_*} & K_0'(X) & \xrightarrow{j^*} & K_0'(U) & \longrightarrow & 0
\end{array}$$

is surjective. Hence, by diagram chase, $v \in \mathrm{im}(\mathrm{res}_X)$.

Let now Y be an arbitrary scheme. Let Z' be an irreducible component in Y and set $Z = f^{-1}Z'$, $U = X - Z$. The number of irreducible components of U is less than one of X. y the first part of the proof and induction hypothesis, the homomorphisms res_Z and res_U in the commutative diagram above are surjective. It follows that res_X is also surjective. □

Corollary 6.3. *Let G be a quasitrivial simply connected group, $T \subset G$ a maximal torus of a Borel subgroup of G. Then the restriction homomorphism*

$$K_0(G/T) = K_0(T; G) \to K_0(G)$$

is surjective.

Proof. The character group T^* is generated by the fundamental characters and therefore, T^* is a permutation Galois module, so that T is a quasisplit torus.

Every principal homogeneous space of T is trivial, hence the statement follows from Theorem 5.9 and Proposition 6.2. □

Theorem 6.4. *Let G be a simply connected group and let X be a principal homogeneous space of G. Then $K_0(X) = \mathbb{Z} \cdot 1$.*

Proof. Suppose first that G is a quasisplit group. Choose a maximal torus T of a Borel subgroup B of G. By Corollary 6.3, the second homomorphism in the composition

$$K_0(G/B) \to K_0(G/T) \to K_0(G)$$

is surjective. The first homomorphism is an isomorphism since every fiber of the canonical surjection $G/T \to G/B$ is isomorphic to an affine space [9, §7, Prop. 4.1]. In the commutative diagram

$$\begin{array}{ccc} K_0(G; G/B) & \longrightarrow & K_0(G; G) = \mathbb{Z} \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ K_0(G/B) & \longrightarrow & K_0(G) \end{array}$$

the left vertical homomorphism is surjective by Corollary 3.2, hence so is the right vertical one, i.e., $K_0(G) = \mathbb{Z} \cdot 1$.

Now let G be an arbitrary simply connected group. Consider the projective homogeneous variety Y of all Borel subgroups of G. For every point $y \in Y$, the group $G_{F(y)}$ is quasisplit. Hence by Proposition 6.2, the homomorphism

$$K_0(Y) \to K_0(G \times Y)$$

is surjective. It follows from Example 2.1 that the natural homomorphism $K_0(F) \to K_0(G)$ is a direct summand of this surjection and therefore, is surjective. □

6.3. **Spectral sequence.** Let G be a split reductive group over a field F. Choose a maximal split torus $T \subset G$. Let X be a G-scheme.

The group $K'_n(G; X)$ (resp. $K'_n(T; X)$) is a module over the representation ring $R(G)$ (resp. $R(T)$). The restriction map $K'_n(G; X) \to K'_n(T; X)$ is a homomorphism of modules with respect to the restriction $R(G) \to R(T)$ and hence induces a $R(T)$-module homomorphism

$$\theta : R(T) \otimes_{R(G)} K'_n(G; X) \to K'_n(T; X).$$

Proposition 6.5. *Assume that the commutator subgroup G' of G is simply connected. Then the homomorphism θ is an isomorphism.*

Proof. Let $B \subset G$ be a Borel subgroup containing T. Set $Y = G/B$. By Proposition 2.2, there is an isomorphism in the category $\mathcal{C}(G)$:

$$F^m \xrightarrow{\sim} Y, \tag{2}$$

defined by elements $u_i \in K_0(G; Y) = R(B)$ forming a base of $R(B)$ over $R(G)$. Applying the realization functor (see Section 2)

$$\mathcal{K}_n^X : \mathcal{C}(G) \to \textbf{Abelian Groups},$$

to the isomorphism (2), we obtain an isomorphism

$$K'_n(G;X)^m \xrightarrow{\sim} K'_n(G;X\times Y).$$

Identifying $K'_n(G;X)^m$ with $R(B)\otimes_{R(G)} K'_n(G;X)$ using the same elements u_i we get the canonical isomorphism

$$R(B)\otimes_{R(G)} K'_n(G;X) \xrightarrow{\sim} K'_n(G;X\times Y).$$

Composing this isomorphism with the canonical isomorphism (Proposition 1.11)

$$K'_n(G;X\times Y) \xrightarrow{\sim} K'_n(B;X),$$

and identifying $K'_n(B;X)$ with $K'_n(T;X)$ via the restriction homomorphism (Theorem 5.1) we get θ.

Since $R(T)$ is free $R(G)$-module by [11, Th.1.3], in the assumptions of Proposition 6.5 one has

(3) $$\operatorname{Tor}_p^{R(G)}(\mathbb{Z}, K'_n(G;X)) \simeq \operatorname{Tor}_p^{R(T)}(\mathbb{Z}, K'_n(T;X)).$$

tell about $R(G)$ somewhere

Thus, Theorem 5.7 yields

Theorem 6.6. *Let G be a split reductive group defined over F with the simply connected commutator subgroup and let X be a G-scheme. Then there exists a spectral sequence*

$$E^2_{p,q} = \operatorname{Tor}_p^{R(G)}(\mathbb{Z}, K'_q(G;X)) \Rightarrow K'_{p+q}(X).$$

`make a proof?`

Remark 6.7. If $X=G$, one has $E^2_{p,q} = \operatorname{Tor}_p^{R(G)}(\mathbb{Z}, K_q(F))$. This formula together with the fact that the spectral sequence degenerates gives the computation of $K_n(G)$ (see [4]).

does it give Panin?

Theorem 5.6 and (3) yield

Corollary 6.8. *If X is a smooth projective G-scheme, then the spectral sequence in Theorem 6.6 degenerates. i.e.,*

$$\operatorname{Tor}_p^{R(G)}(\mathbb{Z}, K'_n(G;X)) = \begin{cases} K'_n(X), & \text{if } p=0;\\ 0, & \text{if } p>0.\end{cases}$$

Example 6.9. Let $H\subset \mathbf{GL}_{n,F}$ be an imbedding. Then for the "classifying variety" $X=\mathbf{GL}_{n,F}/H$ of H one has

$$K_0(X)\simeq \mathbb{Z}\otimes_{R(\mathbf{GL}_{n,F})} R(H).$$

6.4. **Projective case.**

7. Equivariant K-theory of factorial groups

7.1. **Factorial groups.** An algebraic group G over a field F is called *factorial* if for any finite field extension E/F the Picard group $\operatorname{Pic}(G_E)$ is trivial.

Proposition 7.1. [5, Prop. 1.10] *Let G be a reductive group over a field F, G' be the commutator subgroup of G. Then G is factorial if and only if G' is simply connected and the torus G/G' is coflasque.*

In particular, simply connected groups and coflasque tori are factorial.

7.2. **K-theory.**

Theorem 7.2. *Let G be a factorial group and let X be a principal homogeneous space of G. Then $K_0(X) = \mathbb{Z} \cdot 1$.*

Proof. Let G' be the commutator subgroup of G and let $T = G/G'$. The group G' is simply connected and the torus T is coflasque. The variety X is a G'-torsor over $Y = X/G'$. By Proposition 6.2 and Theorem 6.4, the restriction homomorphism

$$K_0(Y) = K_0(G'; X) \to K_0(X)$$

is surjective. The variety Y is a phs of T and by Theorem 5.9, $K_0(Y) = \mathbb{Z} \cdot 1$, whence the result. ☐

Theorem 7.3. *Let G be a reductive group defined over a field F. Then the following condition are equivalent:*
1. *G is factorial.*
2. *For every G-scheme X the restriction homomorphism*

$$K_0'(G; X) \to K_0'(X)$$

is surjective.

Proof. 1 ⇒ 2. Consider first the case when there is a G-torsor $X \to Y$. Then the restriction homomorphism $K_0(G; X) \to K_0(X)$ is surjective by Proposition 6.2 and Theorem 7.2.

In the general case choose a representation $G \to \mathbf{GL}(V)$ such that there is a nonempty open subset $U \subset \mathbb{A}(V)$ on which G acts freely. Consider the commutative diagram

$$\begin{array}{ccccc} K_0'(G; X) & \xrightarrow{\sim} & K_0'(G; \mathbb{A}(V) \times X) & \longrightarrow & K_0'(G; U \times X) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} \\ K_0'(X) & \xrightarrow{\sim} & K_0'(\mathbb{A}(V) \times X) & \longrightarrow & K_0'(U \times X). \end{array}$$

There is a G-torsor $U \times X \to Y$, hence by the first part of the proof, the right vertical map is surjective. By localization, the right horizontal arrows are surjections. Finally, the composition in the bottom row is an isomorphism since it has a splitting $K_0'(U \times X) \to K_0'(X)$ being the inverse image with respect to the closed imbedding $X = \{pt\} \times X \hookrightarrow U \times X$ of finite Tor-dimension (see [9, §7, 2.5]). Thus, the left vertical restriction homomorphism is surjective.

2 ⇒ 1. Taking $X = G_E$ for some finite field extension E/F, we have a surjective homomorphism

$$\mathbb{Z} = K_0(E) = K_0'(G; G_E) \to K_0'(G_E),$$

i.e. $K_0'(G_E) = \mathbb{Z} \cdot 1$. Hence, the first term of the topological filtration $K_0'(G_E)^{(1)}$ of $K_0'(G_E)$ (see [9, §7.5]), being the kernel of the rank homomorphism $K_0'(G_E) \to \mathbb{Z}$, is trivial, in particular,

$$\mathrm{Pic}(G_E) \simeq K_0'(G_E)^{(1/2)} = 0,$$

i.e. G is a factorial group.

7.3. About K_0(*homogeneous*).

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