COHOMOLOGICAL INVARIANTS OF CENTRAL SIMPLE ALGEBRAS

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1. Introduction

Cohomological invariants, introduced by J.-P. Serre in [10], allow us to study algebraic objects by means of Galois cohomology groups.

In this paper we study cohomological invariants of central simple algebras over field extensions of a base field $F$. The tautological degree 2 invariant takes a central simple algebra $A$ over a field $K$ to its class $[A]$ in the Brauer group $\text{Br}(K) = H^2(K, \mathbb{Q}/\mathbb{Z}(1))$. Using cup-products, one can construct invariants of higher degree: if $a \in F^*$, then the cup-product $(a) \cup [A]$ yields a degree 3 invariant of $A$ in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. We call such degree 3 invariants decomposable. Are there indecomposable degree 3 invariants of central simple algebras?

We also study cohomological invariants of tuples of central simple algebras with linear relations in the Brauer group. For example, consider $k$-tuples of quaternion algebras $Q = (Q_1, Q_2, \ldots, Q_k)$ over a field $K$ such that

\[ [Q_1] + [Q_2] + \cdots + [Q_k] = 0 \]

in $\text{Br}(K)$. It turns out that if $k \geq 3$, then there is a nontrivial degree 3 and exponent 2 indecomposable invariant of such tuples defined as follows. Let $\varphi_j$ be the reduced norm quadratic form of $Q_j$. The sum $\varphi$ of the forms $\varphi_j$ in the Witt group $W(K)$ of $K$ belongs to the cube of the fundamental ideal of $W(K)$. The Arason invariant of $\varphi$ in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ yields a nontrivial degree 3 and exponent 2 nontrivial invariant $\text{Ar}_k$ of $k$-tuples $Q$ (see Example 7.2).

Let $n_1, n_2, \ldots, n_k$ be a sequence of positive integers and $D \subset \prod_{j=1}^k (\mathbb{Z}/n_j \mathbb{Z})$ a subgroup. For a field extension $K/F$, let $\text{CSA}_D(K)$ be the set of isomorphism classes of $k$-tuples of central simple $K$-algebras $A = (A_1, A_2, \ldots, A_k)$ with $\deg(A_j) = n_j$ such that $\sum_j d_j[A_j] = 0$ in the Brauer group $\text{Br}(K)$ for all tuples $d = (d_j + n_j \mathbb{Z}) \in D$. Thus, $D$ is the group of relations between the classes of the algebras $A_j$.

Let $d \in D$ be a relation such that $2d_j$ is divisible by $n_j$ for every $j$, i.e., $d$ is of exponent 2 in $D$. For every $A \in \text{CSA}_D(K)$, the class $d_j[A_j]$ in $\text{Br}(K)$ is represented by a quaternion algebra $Q_j$. Then the relation $d$ yields a degree 3 nontrivial indecomposable invariant of $\text{CSA}_D(K)$ taking a tuple $A$ to the cohomology class $\text{Ar}_k(Q)$. In Theorem 7.1, we prove that every degree 3 indecomposable invariant of $\text{CSA}_D$ is of this form and compute the group of all

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invariants. In particular, we show that there are no nontrivial indecomposable invariants of $k$-tuples of simple algebras with relations for $k \leq 2$ and there are no nontrivial indecomposable invariants of CSA$_D$ of odd exponent.

This result is similar to the one on the invariants of étale algebras: Serre proved (see [6, Part 1, Chapter VII]) that étale algebras have no cohomological invariants modulo odd primes, but there are nontrivial invariants of exponent 2 (the Stiefel-Whitney classes of the trace form of the algebra).

We use the following approach to the problem. For every group of relations $D$ there is a reductive algebraic group $G_{\text{red}}$ such that the set of isomorphism classes of $G_{\text{red}}$-torsors over an arbitrary field extension $K$ over $F$ is bijective to the set CSA$_D(K)$. We study degree 3 cohomological invariants of CSA$_D$ via the invariants of $G_{\text{red}}$ using earlier results on degree 3 cohomological invariants of algebraic groups.

Note that every split semisimple group of type $A$ (i.e., every connected component of the Dynkin diagram of $G$ is $A_n$ for some $n$) is embedded to a reductive group $G_{\text{red}}$ corresponding to some group of relations $D$. Moreover, the group of invariants of $G_{\text{red}}$ is identified with the subgroup of reductive invariants of $G$ (see Section 3). Thus, we study degree three reductive cohomological invariants of all split semisimple groups of type $A$.

2. Preliminaries

2.1. Symmetric square and Tor groups. Let $A$ be an abelian group. We write $S^2(A)$ for the symmetric square of $A$, the factor group of $A \otimes A$ by the image of $1-\sigma$, where $\sigma : A \otimes A \to A \otimes A$ is the exchange map, $\sigma(a \otimes a') = a' \otimes a$. We write $aa'$ for the image of $a \otimes a'$ in $S^2(A)$.

The polar homomorphism

$$\text{pol}_A : S^2(A) \to A \otimes A$$

is defined by $\text{pol}(aa') = a \otimes a' + a' \otimes a$.

Let $A$ and $B$ be two abelian groups. We write $A*B$ for the group $\text{Tor}^Z_1(A, B)$. The group $(\mathbb{Z}/n\mathbb{Z}) * (\mathbb{Z}/m\mathbb{Z})$ is cyclic of order $\gcd(n, m)$ with a canonical generator. Write $w_n$ for the canonical generator of $(\mathbb{Z}/n\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$. If $a \in A$ and $b \in B$ are two elements of exponent $n$, we write $[a, n, b]$ for the image of $w_n$ under the homomorphism $(\mathbb{Z}/n\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z}) \to A*B$ given by the maps $a : \mathbb{Z}/n\mathbb{Z} \to A$ and $b : \mathbb{Z}/n\mathbb{Z} \to B$. The elements $[a, n, b]$ generate $A*B$ and are subject to the following relations (see [5, §62]):

1. $[a, n, b]$ is bi-additive in $a$ and $b$,
2. $[a, nm, b] = [a, n, mb]$ if $na = 0$ and $nm = 0$.

Let $\tau : A*A \to A*A$ be the exchange map. Write $\Delta^2(A)$ for the factor group of $A*A$ by $\text{Ker}(1-\tau)$ and $\Sigma^2(A)$ for $(A*A)/\text{Im}(1-\tau)$. If $A$ is a cyclic group, we have $\tau = 1$ and $\Delta^2(A) = 0$. Moreover, $\Delta^2(A \oplus B) \simeq \Delta^2(A) \oplus (A*B) \oplus \Delta^2(B)$. It follows that if $A$ is a direct sum of cyclic groups of order $n_1, n_2, \ldots, n_k$, respectively, then $|\Delta^2(A)| = \prod_{i<j} d_{ij}$, where $d_{ij} = \gcd(n_i, n_j)$.

If $A$ is a cyclic group, we have $\Sigma^2(A) = A*A$. 
**Lemma 2.1.** Let \(a\) be an element of prime order \(p\) in an abelian group \(A\). Then \([a, p, a] \notin \text{Im}(1 - \tau)\), i.e., the coset of \([a, p, a]\) in \(\Sigma^2(A)\) is not trivial.

*Proof.* Let \(A'\) be the cyclic subgroup of \(A\) of order \(p\) generated by \(a\). Choose a homomorphism \(f : A \to C\) to a cyclic group \(C\) such that \(f(a) \neq 0\), i.e., the composition \(A' \hookrightarrow A \xrightarrow{f} C\) is injective. Then the composition

\[A' \ast A' = \Sigma^2(A') \to \Sigma^2(A) \to \Sigma^2(C) = C \ast C\]

is injective since \(\text{Tor}\) is a left exact functor. As \([a, p, a]\) is a generator of the cyclic group \(A' \ast A'\) of order \(p\), the class of \([a, p, a]\) in \(\Sigma^2(A)\) is not trivial. \(\square\)

An exact sequence of abelian groups

\[0 \to A \to B \xrightarrow{q} C \to 0\]

(we identify \(A\) with a subgroup of \(B\)) yields a commutative diagram with the exact column

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \Delta^2(C) & \xrightarrow{\alpha} & \Delta^2(C) & \longrightarrow & S^2(B)/S^2(A) & \longrightarrow & S^2(C) & \longrightarrow & 0 \\
\rho & & \downarrow & & \downarrow & & \delta & & \downarrow & & \text{pol}_C \\
0 & \longrightarrow & C \ast C & \xrightarrow{\gamma} & A \otimes C & \longrightarrow & B \otimes C & \longrightarrow & C \otimes C & \longrightarrow & 0 \\
& & & & \Sigma^2(C) & & & & & & \\
\end{array}
\]

where \(\rho = 1 - \tau\), \(\gamma(\varphi(b), n, c) = nb \otimes c\), \(\alpha = \gamma \circ \rho\), \(\beta(a \otimes \varphi(b)) = ab + S^2(A)\) and \(\delta\) is given by the composition \((1_B \otimes \varphi) \circ \text{pol}_B\).

**Lemma 2.2.** If \(C\) is finite and \(B\) is a free abelian group of finite rank, then the two rows of the diagram are exact.

*Proof.* The lower sequence is exact since \(B\) is free. The map \(\alpha\) is injective since so are \(\gamma\) and \(\rho\). The map \(C \otimes C \to \text{Coker}(\beta)\) taking \(\varphi(b) \otimes \varphi(b')\) to the coset of \(bb'\) yields an inverse of the map \(\text{Coker}(\beta) \to S^2(C)\), whence the exactness of the top row in the term \(S^2(B)/S^2(A)\). The top row is a complex that is acyclic in all terms but possibly \(A \otimes C\).

Choose a \(\mathbb{Z}\)-basis \(x_1, x_2, \ldots, x_s\) for \(B\) such that \(n_1 x_1, n_2 x_2, \ldots, n_s x_s\) is a basis for \(A\) for some positive integers \(n_i\). Then \(x_i x_j\) with \(i \leq j\) is a basis for \(S^2(B)\) and \(n_i n_j x_i x_j\) is a basis for \(S^2(A)\). It follows that \(|S^2(B)/S^2(A)| = \prod_{i < j} d_{ij}\). We also have \(|\Delta^2(A)| = \prod_{i < j} d_{ij}\), where \(d_{ij} = \gcd(n_i, n_j)\) and \(|A \otimes C| = \prod_{i < j} d_{ij}\). A calculation implies that \(|\Delta^2(A)| \cdot |S^2(B)/S^2(A)| = |A \otimes C| \cdot |S^2(C)|\). This proves the exactness of the top row in the term \(A \otimes C\). \(\square\)

Suppose the conditions of Lemma 2.2 hold and we are given an element \(q \in S^2(B)\) and we would like to know the conditions under which \(q \in S^2(A)\). Clearly, the image of \(q\) in \(S^2(C)\) should be trivial and \(\delta(q)\) should be zero.
If these two conditions hold, a diagram chase yields a unique element $\varepsilon(q)$ in $\Sigma^2(C)$ such that $q \in S^2(A)$ if and only if $\varepsilon(q) = 0$.

**Example 2.3.** Let $B = \mathbb{Z}^n/\mathbb{Z}$ with $\mathbb{Z}$ embedded diagonally into $\mathbb{Z}^n$. Write $x_1, \ldots, x_n$ for the canonical generators for $B$, so that $x_1 + \cdots + x_n = 0$. Consider the quadratic form $q = -\sum_{i<j} x_i x_j \in S^2(B)$. Note that $2q = \sum_i x_i^2$. Let $A \subset B$ be a subgroup containing $x_i - x_j$ for all $i$ and $j$ and set $C = B/A$. Write $\bar{x}$ for the common image of the $x_i$’s in $C$, so $C$ is a cyclic group of exponent $n$ generated by $\bar{x}$. Since $\sum_i(x_i - x_1)^2 = 2q + nx_i^2$ and $x_i - x_1 \in A$, we have $2q = -nx_i^2$ in $S^2(B)/S^2(A)$. Therefore, $\beta(nx_1 \otimes \bar{x}) = nx_1^2 = -2q$ and $nx_1 \otimes \bar{x} = \gamma[\bar{x}, n, \bar{x}]$. Thus, $\varepsilon(2q) = -[\bar{x}, n, \bar{x}]$ in $\Sigma^2(C)$.

Let $A$ be an additively written abelian group. We write elements of the group ring $\mathbb{Z}[A]$ in the exponential form: $u = \sum_i r_i e^{a_i}$ for $r_i \in \mathbb{Z}$ and $a_i \in A$. The rank $\operatorname{rank}(u)$ of $u$ is $\sum_i r_i$.

### 2.2. Chern classes.

Let $A$ be a lattice. There are (abstract) Chern class maps (see [8, §3c])

$$c_i : \mathbb{Z}[A] \rightarrow S^i(A).$$

The first Chern class $c_1 : \mathbb{Z}[A] \rightarrow A$ is a homomorphism, $c_1(\sum_i r_i e^{a_i}) = \sum_i r_i a_i$.

The second Chern class satisfies

$$c_2\left(\sum_i e^{a_i}\right) = \sum_{i<j} a_i a_j.$$

and $c_2(u + v) = c_2(u) + c_1(u)c_1(v) + c_2(v)$.

Suppose $A$ is a $W$-module for a finite group $W$. Then the Chern classes are $W$-equivariant. If $a \in A$, we write $We^a$ for the sum $e^{a_1} + e^{a_2} + \cdots + e^{a_k}$ in $\mathbb{Z}[A]$, where $\{a_1, a_2, \ldots, a_k\}$ is the $W$-orbit of $a$. Then $We^a \in \mathbb{Z}[A]^W$ and

$$c_2(We^a) = \sum_{i<j} a_i a_j.$$

We write $\operatorname{Dec}(A)$ for the subgroup of $S^2(A)^W$ generated by $c_2(\mathbb{Z}[A]^W)$. The group $\operatorname{Dec}(A)$ is generated by elements of the following types (see [7, §5]):

1) $\sum_{i<j} a_i a_j$, where $\{a_i\}$ is the $W$-orbit of an element in $A$,

2) $aa'$, where $a, a' \in A^W$.

Thus, $\operatorname{Dec}(G)$ is the subgroup of the “obvious” elements in $S^2(A)^W$.

If $A^W = 0$, the first Chern class is trivial on $\mathbb{Z}[A]^W$, hence the restriction $\mathbb{Z}[A]^W \rightarrow S^2(A)^W$ of $c_2$ is a homomorphism. We will use the following formula proved in [6, §10.14].

**Lemma 2.4.** If $A^W = 0$, we have $c_2(uv) = c_2(u) \operatorname{rank}(v) + c_2(v) \operatorname{rank}(u)$ for all $u, v \in \mathbb{Z}[A]^W$. 
3. Cohomological invariants

Let $\Phi : \text{Fields}_F \to \text{PSets}$ be a functor, where $\text{Fields}_F$ is the category of field extensions of $F$, and $\text{PSets}$ is the category of pointed sets. Let $n$ and $j$ be two integers. For a field extension $K/F$, write $H^n(K, \mathbb{Q}/\mathbb{Z}(j))$ for the Galois cohomology group of the absolute Galois group of $K$ with values in $\mathbb{Q}/\mathbb{Z}(j)$. If $p \neq \text{char}(F)$, the $p$-primary component of $\mathbb{Q}/\mathbb{Z}(j)$ is defined as the colimit over $n$ of the twisted groups of roots of unity $\mu_p^n$. If $p = \text{char}(F) > 0$, then the definition of the $p$-primary component of the cohomology group $H^n(K, \mathbb{Q}/\mathbb{Z}(j))$ requires special care (e.g., see [3, Theorem 5.1]).

A normalized degree $n$ cohomological invariant of $\Phi$ with coefficients in $\mathbb{Q}/\mathbb{Z}(j)$ is collection of maps of pointed sets

$$\Phi(K) \to H^n(K, \mathbb{Q}/\mathbb{Z}(j))$$

for all field extensions $K/F$, natural in $K$, i.e., an invariant is a morphism of functors $\Phi \to H^n(-, \mathbb{Q}/\mathbb{Z}(j))$. We write $\text{Inv}^n(\Phi, \mathbb{Q}/\mathbb{Z}(j))$ for the group of all normalized cohomological invariants of $\Phi$ of degree $n$ with coefficients in $\mathbb{Q}/\mathbb{Z}(j)$.

The cup-product in cohomology yields a pairing

$$F^\times \otimes \text{Inv}^{n-1}(\Phi, \mathbb{Q}/\mathbb{Z}(j-1)) \to \text{Inv}^n(\Phi, \mathbb{Q}/\mathbb{Z}(j)).$$

The cokernel $\text{Inv}^n(\Phi, \mathbb{Q}/\mathbb{Z}(j))_{\text{ind}}$ of this pairing is the group of indecomposable invariants.

Let $G$ be a linear algebraic group over a field $F$ and $\Phi_G$ is the functor taking a field $K$ to the set $H^1(K, G)$ of isomorphism classes of principal homogeneous $G$-spaces ($G$-torsors) over $K$. A cohomological invariant of $\Phi_G$ is also called an invariant of $G$.

We write $\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ for $\text{Inv}^n(\Phi_G, \mathbb{Q}/\mathbb{Z}(j))$ and $\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))_{\text{ind}}$ for $\text{Inv}^n(\Phi_G, \mathbb{Q}/\mathbb{Z}(j))_{\text{ind}}$. By [3, Theorem 2.4], the group $\text{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))$ is isomorphic to $\text{Pic}(G)$ if $G$ is reductive. In this paper we consider cohomological invariants of degree 3. The group of degree 3 indecomposable invariants of split reductive groups was computed in [7, Theorem 5.1.1]:

**Theorem 3.1.** Let $G_{\text{red}}$ be a split reductive group, $T \subset G_{\text{red}}$ a split maximal torus. Then there is an isomorphism

$$\text{Inv}^3(G_{\text{red}}, \mathbb{Q}/\mathbb{Z}(2))_{\text{ind}} \simeq S^2(T^*)^W / \text{Dec}(T^*),$$

where $W$ is the Weyl group of $G$ and $\text{Dec}(T^*)$ is the subgroup of $S^2(T^*)^W$ defined in Section 2.2.

Let $G$ be a split semisimple group over a field $F$, $S \subset G$ a split maximal torus. A reductive envelope of $G$ is a split reductive group $G_{\text{red}}$ over $F$ with the commutator subgroup $G$. Choose a split maximal $T \subset G_{\text{red}}$ such that $T \cap G = S$. We have a natural homomorphism

$$\varphi : S^2(T^*)^W \to S^2(S^*)^W.$$
A reductive envelope $G_{\text{red}}$ of $G$ is called \textit{strict} if the center of $G_{\text{red}}$ is a torus (see [9, Section 9]). If $G_{\text{red}}$ is strict, the image of $\varphi$ is the smallest possible and it is independent of the choice of the strict envelope $G_{\text{red}}$. We write $S^2(S^*)_{\text{red}}$ for $\text{Im}(\varphi)$.

We have the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}[T^*]^W & \longrightarrow & \mathbb{Z}[S^*]^W \\
\downarrow c_2 & & \downarrow c_2 \\
S^2(T^*)^W & \varphi & S^2(S^*)^W.
\end{array}
$$

The top homomorphism in the diagram is surjective (see the proof of [9, Lemma 5.2]). Hence, we have

$\text{Dec}(S^*) \subset S^2(S^*)_{\text{red}} \subset S^2(S^*)^W$.

By [9, Proposition 6.1], the restriction homomorphism

$$\text{Inv}^3(G_{\text{red}}, \mathbb{Q}/\mathbb{Z}(2))_{\text{ind}} \rightarrow \text{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{ind}}$$

is injective (see [9]). Its image is the subgroup $\text{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{red}}$ of \textit{reductive} invariants. Thus, the reductive invariants of $G$ are those indecomposable invariants of $G$ that can be extended to indecomposable invariants of a strict envelope $G_{\text{red}}$ of $G$.

Theorem 3.1 identifies $\text{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{red}}$ with the subgroup $S^2(S^*)_{\text{red}}/\text{Dec}(S^*)$ of $S^2(S^*)^W / \text{Dec}(S^*)$.

Let $G$ be a split semisimple simply connected group over $F$. Then $G = G_1 \times G_2 \times \cdots \times G_k$, where $G_j$ are (almost) simple simply connected groups. Let $S_j \subset G_j$ be a maximal torus, $W_j$ the Weyl group of $G_j$. Then $S = S_1 \times S_2 \times \cdots \times S_k$ is a maximal torus of $G$ and $W = W_1 \times W_2 \times \cdots \times W_k$ is the Weyl group of $G$.

The group $S^2(S^*)^W$ can be viewed as the group of $W$-invariant integer quadratic forms on the lattice of co-characters $S_*$. The group $S^2(S^*)^W$ is free with a canonical basis $q_1, q_2, \ldots, q_k$, where $q_j$ is (the only) $W_j$-invariant quadratic form on $(S_j)_*$ that has value 1 on a short co-root of $G_j$ (see [6, Part 2, §10]).

We have the second Chern class homomorphism (note that $(S^*)^W = 0$)

$$c_2 : \mathbb{Z}[S^*]^W \rightarrow S^2(S^*)^W.$$

If $u \in \mathbb{Z}[S^*]^W$, we write

$$(1) \quad c_2(u) = N_1(u)q_1 + N_2(u)q_2 + \cdots + N_k(u)q_k$$

with unique $N_j(u) \in \mathbb{Z}$.

4. \textbf{Central simple algebras with relations}

Let $n_1, n_2, \ldots, n_k$ be positive integers and $D$ a subgroup of $\prod_{j=1}^k(\mathbb{Z}/n_j\mathbb{Z})$. Consider a functor

$$\text{CSA}_D : \text{Fields}_F \rightarrow \text{PSets}$$
that takes a field extension $K/F$ to the set $\text{CSA}_D(K)$ of $k$-tuples of central simple $K$-algebras $(A_1, A_2, \ldots, A_k)$ with $\deg(A_j) = n_j$ such that $\sum_j d_j[A_j] = 0$ in the Brauer group $\text{Br}(K)$ for all tuples $(d_j + n_j\mathbb{Z}) \in D$. We call $D$ the group of relations between classes of central simple algebras.

We show that the functor $\Phi$ is isomorphic to the functor $\Phi_{G_{\text{red}}}$ for a reductive group $G_{\text{red}}$. The group $\prod_{j=1}^k (\mathbb{Z}/n_j\mathbb{Z})$ is the character group of $\mu := \prod_{j=1}^k \mu_{n_j}$.

Let $\mathcal{Z} \subseteq \mu$ be a subgroup such that $\mathcal{Z}^* = \mu^*/D$. Let $\mathcal{Z}$ be a subgroup such that $\mathcal{Z}^* = \mu^*/D$.

Write $G$ for the factor group of the product $\prod_{j=1}^k \text{SL}_{n_j}$ by $\mathcal{Z}$ and set $G_{\text{red}} = (\prod_{j=1}^k \text{GL}_{n_j})/\mathcal{Z}$. Then $G_{\text{red}}$ is a strict envelope of $G$. Note that $D$ is naturally isomorphic to the character group of the center of $G$.

The natural surjection $G_{\text{red}} \to \prod_{j=1}^k \text{PGL}_{n_j}$ yields a map

$$\rho : H^1(K, G_{\text{red}}) \to \prod_{j=1}^k H^1(K, \text{PGL}_{n_j})$$

for every field extension $K/F$. Recall that the set $H^1(K, \text{PGL}_{n_j})$ is naturally bijective to the set of isomorphism classes of central simple algebras of degree $n_j$. Therefore, a $G_{\text{red}}$-torsor over $K$ yields a tuple of central simple $K$-algebras $(A_1, A_2, \ldots, A_k)$ with $\deg(A_j) = n_j$.

**Proposition 4.1.** [4, Theorem A1] The map $\rho$ establishes a bijection between $\Phi_{G_{\text{red}}}(K) = H^1(K, G_{\text{red}})$ and the set $\text{CSA}_D(K)$ for every field extension $K/F$.

The group of invariants $\text{Inv}^n(G_{\text{red}}, \mathbb{Q}/\mathbb{Z}(j))$ is identified with the subgroup of reductive invariants $\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ in $\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$. Thus, we can view $\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))_{\text{red}}$ as the group of cohomological invariants of the set of $k$-tuples of central simple algebras of given degrees $n_j$ and satisfying linear relations given by the group of relations $D$.

5. Simple groups of type $A$

5.1. Case $G = \text{SL}_n$. Write $B$ for the character group of the maximal torus of diagonal matrices. Then $B = \mathbb{Z}^n/\mathbb{Z} = \sum_i \mathbb{Z} x_i$ (see Example 2.3) and $B$ is the weight lattice of the root system $A_{n-1}$. The root sublattice $\Lambda_r \subset B$ is generated by roots $x_i - x_j$. The Weyl group $W$ is the symmetric group $S_n$ acting by permutations on the $x_i$'s. The factor group $B/\Lambda_r$ is equal to $(\mathbb{Z}/n\mathbb{Z})^{\hat{x}}$, where $\hat{x}$ is the class of $x_i$ (it is independent of $i$). For every character $\gamma \in B$ we write $\hat{\gamma} = a\hat{x}$ for its residue in $(\mathbb{Z}/n\mathbb{Z})^{\hat{x}}$.

Choose a character $\gamma \in B$. Some of the components of $\gamma$ may coincide. Let $\gamma$ have distinct components $a_1 > a_2 > \cdots > a_k$ which repeat $r_1, r_2, \ldots, r_k$ times respectively, so that $n = \sum_i r_i$ and $\hat{\gamma} = a\hat{x}$ with $a = \sum_i r_i a_i$. We denote the character $\gamma$ by $(r_1, \ldots, r_k; a_1, \ldots, a_k)$ or simply by $(r, a)$ (see [6, Part 2, §11]).

The stabilizer of $\gamma = (r, a)$ in the Weyl group $W = S_n$ is isomorphic to the product $S_{r_1} \times S_{r_2} \times \cdots \times S_{r_k}$ of symmetric groups. Hence the rank of $W e'$,
Let $v^p$ be the $p$-adic valuation for a prime $p$.

**Lemma 5.1.** Let $\hat{y} = a\hat{x}$ for $y \in B$. Then $v_p(\text{rank}(W^y)) \geq v_p(n) - v_p(a)$ for every prime $p$.

*Proof.* Write $y = (r, a)$ as above. Let $l = \min_i v_p(r_i)$. Since $a \equiv \sum_i r_i a_i$ modulo $n$ and $n = \sum_i r_i \in p^l \mathbb{Z}$, we have $v_p(a) \geq l$. By [6, Lemma 11.3],

$$v_p(n! / r_1! r_2! \cdots r_k!) \geq v_p(n) - l.$$

The result follows from (2). \hfill \Box

Recall that $c_2(W^y) = N(W^y)q$, where $q = -\sum_{i<j} x_i x_j \in S^2(B)W$ (see (1)).

**Lemma 5.2.** Let $y \in B$ be such that $\hat{y} = a\hat{x}$ with $v_p(a) \leq v_p(n)$, then $v_p(N(W^y)) \geq v_p(a)$.

*Proof.* Write $y = (r, a)$. By [6, Lemma 11.4], the gcd of $\sum_i r_i a_i$ and $n$ divides $v_p(N(W^y))$. Since $a \equiv \sum_i r_i a_i$ modulo $n$, the result follows from the assumption on $a$. \hfill \Box

The following statement shows that the inequalities in Lemmas 5.1 and 5.2 are sharp.

**Lemma 5.3.** Let $a$ be an integer with $v_p(a) < v_p(n)$ for a prime $p$. Then there is a character $y \in B$ such that

1. $\hat{y} = a\hat{x}$ in $(\mathbb{Z}/n\mathbb{Z})\hat{x}$,
2. $v_p(\text{rank}(W^y)) = v_p(n) - v_p(a)$,
3. $v_p(c_2(W^y)) = v_p(a)$.

*Proof.* Write $a = p^n v$ for an integer $v$ prime to $p$ and $u = v_p(a)$. Consider the character $z = x_1 + x_2 + \cdots + x_{p^n} \in B$. By [2, Section 4.2], we have $v_p(c_2(W^z)) = v_p(a)$. If $y := vz$ then $\hat{y} = v^n a\hat{x} = a\hat{x}$ and $c_2(W^y) = v^2 c_2(W^z)$, hence $v_p(c_2(W^y)) = v_p(c_2(W^z)) = v_p(a)$. Finally, $\text{rank}(W^z) = \binom{n}{p^n}$ and

$$v_p(\text{rank}(W^y)) = v_p(\text{rank}(W^z)) = v_p(n) - u = v_p(n) - v_p(a).$$

\hfill \Box

### 5.2. Case $G = \text{SL}_n / \mu_m$. Let $m$ be a divisor of $n$ and set $G = \text{SL}_n / \mu_m$. Let $A \subset B = \mathbb{Z}^n / \mathbb{Z} = \sum_i \mathbb{Z} x_i$ be the character group of the maximal torus $S$ of classes of diagonal matrices. Thus $A$ is the subgroup of $B$ containing the root lattice $\Lambda_r$. The factor group $C = B/A = (\mu_m)^n$ is equal to $(\mathbb{Z}/m\mathbb{Z})\hat{x}$, where $\hat{x}$ is the coset $x_i + A$ in $C$. The Weyl group $W$ trivially on $C$, hence $A$ is a $W$-submodule of $B$. We have the following groups:

$$\text{Dec}(A) \subset S^2(A)_{\text{red}} \subset S^2(A)W \subset S^2(B)W = \mathbb{Z}q,$$
where $q = - \sum_{i<j} x_i x_j \in S^2(B)^W$.

**Lemma 5.4.** If $kq \in S^2(A)^W_{\text{red}}$, then $k \in m\mathbb{Z}$.

**Proof.** The class $\bar{x}$ in $C = (\mathbb{Z}/m\mathbb{Z})\bar{x}$ of first fundamental weight $x_1$ of $G$ has order $m$. By [9, Proposition 10.6] or [7, Proposition 7.1], $k$ is divisible by $m$. $\square$

**Lemma 5.5.** We have $2nq \in \text{Dec}(A)$.

**Proof.** Consider the character $x = x_1 - x_2 \in A$. By [8, Section 4b], $c_2(We^x) = -2nq \in \text{Dec}(A)$. $\square$

**Lemma 5.6.**
1. For every odd prime $p$, there is an integer $k$ prime to $p$ such that $kqm \in \text{Dec}(A)$.
2. Suppose that either $n$ is odd or $v_2(m) < v_2(n)$. Then there is an odd integer $k$ such that $kqm \in \text{Dec}(A)$.

**Proof.** Let $p$ be a prime integer. Suppose first that $v_p(m) < v_p(n)$. Let $r = v_p(m)$. By Lemma 5.3 applied to the integer $a = m$, there is a character $y \in B$ such that $v_p(N(We^y)) = v_p(m)$ and $\bar{y} = m\bar{x} = 0$ in $(\mathbb{Z}/m\mathbb{Z})\bar{x}$. In particular, $y \in A$ and $c_2(We^y) = kmq$ with $k$ prime to $p$.

Now let $p$ be an odd prime with $v_p(m) = v_p(n)$. By Lemma 5.5, $(2n/m)mq \in \text{Dec}(A)$ and $2n/m$ is prime to $p$.

Finally, let $n$ be odd. We have $mx_1 \in A$ and $c_2(We^{mx_1}) = m^2q \in \text{Dec}(A)$ and $m$ is odd as it divides $n$. $\square$

Now we are going to use the invariant $\varepsilon$ defined in Section 2.

**Lemma 5.7.** If $k$ is divisible by $m$ and $v_2(m) = v_2(n) > 0$, we have $\varepsilon(kq) = \left[ \frac{k}{2} \bar{x}, 2, \frac{k}{2} \bar{x} \right]$ in $\Sigma^2(C)$.

**Proof.** Since $n/m$ is odd and $m\bar{x} = 0$, we have by Example 2.3:

$$\varepsilon(mq) = -\frac{m}{2} \left[ \bar{x}, n, \bar{x} \right] = -\left[ \frac{m}{2} \bar{x}, 2, \frac{m}{2} \bar{x} \right] = -\left[ \frac{m}{2} \bar{x}, 2, \frac{m}{2} \bar{x} \right] = \left[ \frac{m}{2} \bar{x}, 2, \frac{m}{2} \bar{x} \right].$$

It follows that $\varepsilon(kq) = \left[ \frac{k}{2} \bar{x}, 2, \frac{k}{2} \bar{x} \right]$ since both sides are equal to $\varepsilon(mq) = \left[ \frac{m}{2} \bar{x}, 2, \frac{m}{2} \bar{x} \right]$ if $k/m$ is odd and is equal to zero if $k/m$ is even. $\square$

**Proposition 5.8.** Let $G = \text{SL}_n / \mu_m$ and $S$ a maximal split torus of $G$. Then

$$\text{Dec}(S^*) = S^2(S^*)_{\text{red}}^W = \begin{cases} 2m\mathbb{Z}q, & \text{if } v_2(m) = v_2(n) > 0; \\ m\mathbb{Z}q, & \text{otherwise}. \end{cases}$$

**Proof.** The second case follows from Lemmas 5.4 and 5.6. Suppose $v_2(m) = v_2(n) > 0$. It follows from Lemmas 5.5 and 5.6 that $2mq \in \text{Dec}(A)$. It suffices to show that if $kq \in S^2(A)^W_{\text{red}}$, then $k \in 2m\mathbb{Z}$. By Lemma 5.4, $k$ is divisible by $m$. Recall that $\bar{x}$ has order $m$ in $C = B/A$. In view of Lemma 5.7, $\varepsilon(kq) = \left[ \frac{k}{2} \bar{x}, 2, \frac{k}{2} \bar{x} \right]$ in $\Sigma^2(C)$. By Lemma 2.1, $\frac{k}{2} \bar{x} = 0$ in $C$, i.e., $k \in 2m\mathbb{Z}$. $\square$
It follows from Proposition 5.8 that every reductive invariant of $\text{SL}_n/\mu_m$ is trivial (see [7, §7]) or, equivalently, central simple algebras of degree $n$ and exponent dividing $m$ have no indecomposable degree 3 invariants.

6. Semisimple groups of type $A$

Let $n_1, n_2, \ldots, n_k$ be positive integers and $D$ a subgroup of relations in $\prod_{j=1}^k (\mathbb{Z}/n_j \mathbb{Z})$. Let $Z \subset \mu$ be the subgroup such that $Z^* = \mu^*/D$ and $G = (\prod_{j=1}^k \text{SL}_{n_j})/\mathbb{Z}$ as in Section 4.

Let $B = B_1 \oplus B_2 \cdots \oplus B_k$ denote the character group of a split maximal torus of $G$ with the $B_j$’s as in Section 5.2. Write $A$ for the kernel of the natural surjection $B \to C =: Z^*$, so $A$ is the character lattice of a split maximal torus of $G$. For every $j$, the image of the projection $Z \to \mu_{n_j}$ is the subgroup $\mu_{m_j}$ of $\mu_{n_j}$ for a divisor $m_j$ of $n_j$. We have then natural homomorphisms $G \to \text{SL}_{n_j}/\mu_{m_j}$. Write $\tilde{x}_j$ for the canonical generator of the cyclic group $(\mu_{m_j})^* \subset B/A = C$ of order $m_j$. Thus, $C$ is generated by the $\tilde{x}_j$’s.

The group $D$ is the kernel of the natural surjection $B/\Lambda_r \to C$, so $D$ is the character group of the center of $G$. We have the following diagram with the exact rows:

$$
\begin{array}{c}
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \\
0 \longrightarrow D \longrightarrow B/\Lambda_r \longrightarrow C \longrightarrow 0.
\end{array}
$$

Note that $B/\Lambda_r = \prod_j (\mathbb{Z}/n_j \mathbb{Z})\tilde{x}_j$, where $\tilde{x}_j$ is the class of a canonical generator of $B_j$ in $B/\Lambda_r$. The image of $\tilde{x}_j$ under the homomorphism $B/\Lambda_r \to C$ is equal to $\tilde{x}_j$.

The Weyl group $W$ of $G$ is the product of symmetric groups $W_j = S_{n_j}$. Write $q_j \in S^2(B_j)^W \subset S^2(B)^W$ for the canonical quadratic forms (see Section 5.2). Then $\{q_1, q_2, \ldots, q_k\}$ is a $\mathbb{Z}$-basis for $S^2(B)^W$.

Below is a generalization of Lemma 5.4.

**Lemma 6.1.** If $\sum_j k_j q_j \in S^2(A)^W_{\text{red}}$, then $k_j \in m_j \mathbb{Z}$ for all $j$.

**Proof.** The class in $C$ of first fundamental weight of the $j$th component of $G$ has order $m_j$. By [9, Proposition 10.6] or [7, Proposition 7.1], $k_j$ is divisible by $m_j$. \qed

Consider the subset $J \subset \{1, 2, \ldots, k\}$ of all $j$ such that $v_2(m_j) = v_2(n_j) > 0$. Write $D'$ for the subgroup of $D$ of all elements having zero components outside $J$, i.e.,

$$
D' = D \cap \prod_{j \in J} (\mathbb{Z}/n_j \mathbb{Z})\tilde{x}_j.
$$

Let $q = \sum_{j \in J} k_j q_j \in S^2(B)^W$ be such that $k_j \in m_j \mathbb{Z}$ for every $j$. By Lemma 5.7,

$$
\varepsilon(q) = \sum_{j \in J} [\tilde{k}_j \tilde{x}_j, 2, \tilde{k}_j \tilde{x}_j] \text{ in } \Sigma^2(C),
$$

where $\varepsilon$ is the basic invariant.
where $\tilde{k}_j = k_j/2$. Let $x \in B$ be a character with $\check{x} := \sum_{j \in J} \tilde{k}_j \check{x}_j \in B/\Lambda_r$. Since

$$[\check{x}, 2, \check{x}] = \sum_{j \in J} [\tilde{k}_j \check{x}_j, 2, \tilde{k}_j \check{x}_j] + \sum_{j \neq i} [\tilde{k}_j \check{x}_j, 2, \tilde{k}_i \check{x}_i]$$

and

$$[\tilde{k}_j \check{x}_j, 2, \tilde{k}_i \check{x}_i] + [\tilde{k}_i \check{x}_i, 2, \tilde{k}_j \check{x}_j] = [\tilde{k}_j \check{x}_j, 2, \tilde{k}_i \check{x}_i] - [\tilde{k}_i \check{x}_i, 2, \tilde{k}_j \check{x}_j] \in \text{Im}(1 - \tau)$$

for $j \neq i$, we have

$$(3) \quad \varepsilon(q) = [\check{x}, 2, \check{x}] \in \Sigma^2(C).$$

**Proposition 6.2.** Let $q = \sum_j k_j q_j \in S^2(B)^W$. The following conditions are equivalent:

1. $q \in S^2(A)^W_{\text{red}}$.
2. $q' := \sum_{j \in J} k_j q_j \in S^2(A)^W_{\text{red}}$ and $k_j \in m_j \mathbb{Z}$ for every $j$,
3. $k_j$ is even for every $j \in J$ and $\sum_{j \in J} k_j \hat{x}_j \in D'$ and $k_j \in m_j \mathbb{Z}$ for all $j$.

**Proof.** Set $\check{x} := \sum_{j \in J} \tilde{k}_j \check{x}_j \in B/\Lambda_r$.

1. $\Rightarrow$ (2): By Lemma 6.1, $k_j \in m_j \mathbb{Z}$ for all $j$. If $j \notin J$, then by Proposition 5.8, $k_j q_j \in S^2(A)^W_{\text{red}} \subset S^2(A)^W_{\text{red}}$. It follows that $q' \in S^2(A)^W_{\text{red}}$.

2. $\Rightarrow$ (3): By (3), $0 = \varepsilon(q') = [\check{x}, 2, \check{x}]$ in $\Sigma^2(C)$. In view of Lemma 2.1, $\check{x} = 0$ in $C$, i.e., $\check{x} \in D'$. Then $\check{x} \in D'$.

3. $\Rightarrow$ (1): We have $\check{x} \in D'$ and $k_j \in m_j \mathbb{Z}$ for all $j \in J$. In particular, $k_j$ is even. It follows from (3) that $\varepsilon(q') = [\check{x}, 2, \check{x}] = 0$ in $\Sigma^2(C)$, hence $q' \in S^2(A)^W_{\text{red}}$. If $j \notin J$, then by Proposition 5.8, $k_j q_j \in S^2(A)^W_{\text{red}}$. Thus, $q \in S^2(A)^W_{\text{red}}$. \hfill $\square$

Consider a homomorphism

$$\alpha : \overline{2D'} \rightarrow S^2(A)^W_{\text{red}} / \text{Dec}(A),$$

where $\overline{2D'}$ is the subgroup of exponent 2 elements in $D'$, defined as follows. Let $x \in \overline{2D'}$, i.e., $\check{x} := \sum_{j \in J} k_j \hat{x}_j$ with $k_j \in n_j \mathbb{Z}$. Set

$$\alpha(\check{x}) = \sum_{j \in J} k_j q_j + \text{Dec}(A).$$

We have $\alpha$ well defined by Proposition 6.2.

**Lemma 6.3.** There are no elements in $\overline{2D'}$ with exactly one nonzero component.

**Proof.** Suppose that $\frac{n_j}{2} \hat{x}_j \in \overline{2D'}$ for some $j \in J$. Then $\frac{n_j}{2} \check{x}_j = 0$ in $C$. It follows that $m_j$ divides $\frac{n_j}{2}$ since the order of $\check{x}_j$ in $C$ is equal to $m_j$. This is a contradiction since $v_2(m_j) = v_2(n_j)$ for $j$ in $J$. \hfill $\square$

Let $E$ be the subgroup of $\overline{2D'}$ generated by all elements with exactly two nonzero components.

**Lemma 6.4.** We have $\alpha(E) = 0$. 
Proof. Every generator of $E$ is of the form $\frac{n_j}{2}\hat{x}_j + \frac{n_k}{2}\hat{x}_k$ with $j \neq k$ in $J$. We want to show that $n_jq_j + n_kq_k \in \text{Dec}(A)$. By Lemma 5.3 applied to the integers $\frac{n_j}{2}$ and $\frac{n_k}{2}$, respectively, there are characters $y_j \in B_j$ and $y_k \in B_k$ such that

(1) $\hat{y}_j = \frac{n_j}{2}\hat{x}_j$ in $(\mathbb{Z}/n_j\mathbb{Z})\hat{x}_j$, $\hat{y}_k = \frac{n_k}{2}\hat{x}_k$ in $(\mathbb{Z}/n_k\mathbb{Z})\hat{x}_k$,

(2) $v_2(\text{rank}(W_j e^{y_j})) = 1$, $v_2(\text{rank}(W_k e^{y_k})) = 1$,

(3) $v_2(N(W_j e^{y_j})) = v_2(n_j) - 1$, $v_2(N(W_k e^{y_k})) = v_2(n_k) - 1$.

Set $y := y_j + y_k$. As $\hat{y} = \hat{y}_j + \hat{y}_k \in E \subset 2D'$, we have $y \in A$. It follows from the equality

$$W e^y = W_j e^{y_j} \cdot W_k e^{y_k}$$

and Lemma 2.4 that

$$c_2(W e^y) = c_2(W_j e^{y_j} \cdot W_k e^{y_k}) = c_2(W_j e^{y_j}) \text{rank}(W_k e^{y_k}) + c_2(W_k e^{y_k}) \text{rank}(W_j e^{y_j})$$

$$= N(W_j e^{y_j}) \text{rank}(W_k e^{y_k})q_j + N(W_k e^{y_k}) \text{rank}(W_j e^{y_j})q_k$$

$$= t_jq_j + t_kq_k$$

for the integers $t_j$ and $t_k$ with $v_2(t_j) = v_2(n_j)$ and $v_2(t_k) = v_2(n_k)$. Recall that $2n_jq_j$ and $2n_kq_k$ belong to $\text{Dec}(A)$ by Lemma 5.5. It follows that $n_jq_j + n_kq_k \in \text{Dec}(A)$.

It follows from Lemma 6.4 that $\alpha$ factors through a homomorphism

$$\alpha' : (2D')/E \to S^2(A)_{\text{red}}/\text{Dec}(A).$$

We prove that $\alpha'$ is an isomorphism by constructing the inverse map. Define a homomorphism

$$\beta : S^2(A)_{\text{red}} \to 2D'$$

as follows. Let $q = \sum j k_j q_j \in S^2(A)_{\text{red}}$. By Lemma 6.1, $k_j \in m_j\mathbb{Z}$ for all $j$. Set

$$\beta(q) = \sum_{j \in J} \frac{k_j n_j}{2m_j} \hat{x}_j.$$

By Proposition 6.2, $\sum_{j \in J} \frac{k_j}{2} \hat{x}_j \in D'$. Since $m_j\hat{x}_j \in D'$ and $n_j/m_j$ is odd, we have $\beta(q) \in D'$. Also, $2\beta(q) = 0$ since $n_j\hat{x}_j = 0$, hence $\beta(q) \in 2D'$.

Lemma 6.5. We have $\beta(\text{Dec}(A)) \subset E$.

Proof. We shall show that $\beta(c_2(W e^y)) \in E$ for every $y \in A$. Write $\hat{y} = \sum a_j \hat{x}_j$ for some $a_j \in \mathbb{Z}$ (unique modulo $n_j$). Since $c_2(W e^y) = t^2 c_2(W e^y)$ for every integer $t$, we may replace $y$ by $ty$ for every odd integer $t$. In particular, we may assume that either $a_j = 0$ or $v_2(a_j) < v_2(n_j)$ for every $j$.

Let $s$ be the number of indices $j$ such that $a_j \neq 0$.

Case 1: $s \leq 2$. In this case $c_2(W e^y)$ has at most 2 nonzero $j$-components, hence $\beta(c_2(W e^y)) \in E$.

Case 2: $s \geq 3$. We show that $\beta(c_2(W e^y)) = 0$. Fix a $k \in J$. It suffices to prove that $v_2$ of the $q_k$-coefficient $N_k(W e^y)$ of $c_2(W e^y)$ is strictly larger than $v_2(m_k) = v_2(n_k)$. Set $t_j := v_2(n_j) - v_2(a_j)$ for all $j$ such that $a_j \neq 0$. 

We claim that there is an \(i\) different from \(k\) such that
\[
(4) \quad t_i \geq t_k.
\]
Suppose that \(t_k > t_i\) for all \(i\) different from \(k\). Then there is an odd integer \(s\) such that \(s2^k t_k = s2^k \tilde{x}_k\) is a nonzero element in \(2D'\) with only one nonzero component, a contradiction by Lemma 6.3. The claim is proved.

Write \(y = \sum_j y_j\), where \(y_j \in B_j\). We have \(\tilde{y}_j = a_j \tilde{x}_j\) for all \(j\) and
\[
(5) \quad W e^y = \prod_j W_j e^{y_j} = W_k e^{y_k} \cdot z,
\]
where \(z\) is the product of all \(W_j e^{y_j}\) but \(W_k e^{y_k}\). Hence by Lemma 2.4,
\[
c_2(W e^y) = N_k(W_k e^{y_k}) \cdot \text{rank}(z) q_k + (\text{linear combination of} q_j\text{'s with} j \neq k).
\]
By Lemma 5.2,
\[
(6) \quad \nu_2(N_k(W_k e^{y_k})) \geq \nu_2(a_k).
\]
Also, \(z\) is divisible by \(W_i e^{y_i} W_j e^{y_j}\) for \(i\) as in (4) and some \(j\) such that \(a_j \neq 0\) (such exists since \(s \geq 3\)). We have then
\[
(7) \quad \text{rank}(z) \in \text{rank}(W_i e^{y_i}) \cdot \text{rank}(W_j e^{y_j}) Z.
\]
By Lemma 5.1,
\[
(8) \quad \nu_2(\text{rank}(W_i e^{y_i})) \geq \nu_2(n_i) - \nu_2(a_i) = t_i
\]
and
\[
(9) \quad \nu_2(\text{rank}(W_j e^{y_j})) \geq \nu_2(n_j) - \nu_2(a_j) > 0.
\]
It follows from (4)–(9) that
\[
\nu_2(N_k(W e^y)) = \nu_2(N_k(W_k e^{y_k})) + \nu_2(\text{rank}(z)) \geq \nu_2(N_k(W_k e^{y_k})) + \nu_2(\text{rank}(c_2(W e^{y_i}))) + \nu_2(\text{rank}(c_2(W_j e^{y_j})))
\]
\[
> \nu_2(a_k) + t_i
\]
\[
\geq \nu_2(a_k) + t_k
\]
\[
= \nu_2(n_k).
\]
It follows from Lemma 6.5 that \(\beta\) factors through a homomorphism
\[
\beta' : S^2(A)^W_{\text{red}} / \text{Dec}(A) \to (2D') / E.
\]

**Proposition 6.6.** Let \(S\) be a maximal split torus of the group \(G = (\prod_{j=1}^k \text{SL}_{m_j}) / Z\). Then the map \(\alpha' : (2D') / E \to S^2(S^*)^W_{\text{red}} / \text{Dec}(S^*)\) is an isomorphism.

**Proof.** We show that \(\beta'\) is the inverse of \(\alpha'\). The composition \(\beta' \circ \alpha'\) is the identity since \(n_j / m_j\) is odd for all \(j \in J\). Let \(q = \sum_j k_j q_j \in S^2(A)^W_{\text{red}}\). By Lemma 6.1, \(k_j \in m_j Z\) for all \(j\). We have \(\alpha' \circ \beta'(q) = \sum_{j \in J} k_{n_j} m_j q_j\). It follows from Proposition 5.8 that \(2k_j q_j \in \text{Dec}(A)\) for \(j \in J\), therefore, \(k_{n_j} m_j q_j\) is congruent to \(k_j q_j\) modulo \(\text{Dec}(A)\) since \(n_j / m_j\) is odd.
If \( j \notin J \), then by Proposition 5.8, \( k_j q_j \in \text{Dec}(A) \). It follows that \( \alpha' \circ \beta'(q) \)
is equal to \( q \) modulo \( \text{Dec}(A) \). \( \Box 

7. Main theorem

Let \( n_1, n_2, \ldots, n_k \) be a sequence of positive integers, \( D \subseteq \coprod_{j=1}^{k} (\mathbb{Z}/n_j \mathbb{Z}) \) a subgroup of relations. Let CSA\(_D\) be the functor that takes a field extension \( K/F \) to the set of \( k \)-tuples of central simple \( K \)-algebras \((A_1, A_2, \ldots, A_k)\) such that \( \sum_j d_j \cdot [A_j] = 0 \) in the Brauer group \( \text{Br}(K) \) for all tuples \((d_1 + n_1 \mathbb{Z}, \ldots, d_k + n_k \mathbb{Z}) \in D \).

For every \( j \), write \( m_j \mathbb{Z} / n_j \mathbb{Z} = D \cap (\mathbb{Z}/n_j \mathbb{Z}) \) for a unique positive divisor \( m_j \) of \( n_j \). Consider the set \( J \) of all indices \( j \) such that \( v_2(m_j) = v_2(n_j) > 0 \) and let \( D' = D \cap \coprod_{j \in J} (\mathbb{Z}/n_j \mathbb{Z}) \). Let \( E \) be the subgroup of \( 2D' \) generated by elements with exactly two nonzero components.

Combining Theorem 3.1 and Propositions 4.1 and 6.6, we get the following main theorem of the paper.

**Theorem 7.1.** For every group of relations \( D \), there is a natural isomorphism 
\( (2D')/E \overset{\sim}{\longrightarrow} \text{Inv}^3(\text{CSA}_D, \mathbb{Q}/\mathbb{Z}(2))_{\text{ind}} \).

**Example 7.2.** Let \( n_1 = n_2 = \cdots = n_k = 2 \) for \( k \geq 3 \) and let \( D \) be the cyclic subgroup (of order 2) generated by \((1, 1, \ldots, 1)\). Then CSA\(_D\)(\( K \)) is the set of \( k \)-tuples of quaternion \( K \)-algebras \((Q_1, Q_2, \ldots, Q_k)\) such that 
\[ [Q_1] + [Q_2] + \cdots + [Q_k] = 0 \]
in \( \text{Br}(K) \). We have \( 2D' = D = \mathbb{Z}/2\mathbb{Z} \) and \( E = 0 \), i.e., there is exactly one indecomposable degree 3 invariant of CSA\(_D\). It is defined as follows (see [9, Example 11.2]). Let \( \varphi_j \) be the reduced norm quadratic form of \( Q_j \). The sum \( \varphi \) of the forms \( \varphi_j \) in the Witt group \( W(K) \) of \( K \) belongs to the cube of the fundamental ideal of \( W(K) \) (this also makes sense when \( \text{char}(F) = 2 \)), i.e., \( \varphi \) is the sum of 3-fold Prister forms \( \rho_1, \rho_2, \ldots, \rho_s \). The Arason invariant \( \sum_i e_3(\rho_i) \) of \( \varphi \) in \( H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \), where \( e_3(\rho_i) \) is the class of \( \rho_i \) in \( H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \), yields the only nontrivial degree 3 nontrivial invariant \( \text{Ar}_k \) of CSA\(_D\) (see also [1]).

We can make explicit the isomorphism in Theorem 7.1. Let \( d \in 2D' \). Write \( d = \sum_j d_j \hat{x}_j \) for integers \( d_j \) such that \( 2d_j \in n_j \mathbb{Z} \). The map of \( (\mathbb{Z}/2\mathbb{Z})^k \) to \( \coprod_{j=1}^{k} (\mathbb{Z}/n_j \mathbb{Z}) \hat{x}_j \) taking a tuple \((b_j)\) to \( \sum_j b_j d_j \hat{x}_j \) sends the generator \((1, 1, \ldots, 1)\) from Example 7.2 to \( d \). This describes the invariant \( P_d \) of CSA\(_D\) corresponding to \( d \) by Theorem 7.1 as follows. Let \( A = (A_1, A_2, \ldots, A_k) \) be a tuple of central simple algebras in CSA\(_D\)(\( K \)). In particular, \( \sum_j d_j \cdot [A_j] = 0 \) in \( \text{Br}(K) \). As \( \deg(A_j) = n_j \), the class \( d_j \cdot [A_j] \) is represented by a quaternion algebra \( Q_j \), and we have \( \sum_j [Q_j] = 0 \). The invariant \( P_d \) is given by \( P_d(A) := \text{Ar}_k(Q) \), where \( Q = (Q_j) \) with \( \text{Ar}_k \) from Example 7.2.

**References**


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