Equivariant K-theory

V.2

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Introduction

2.1

The equivariant K-theory was developed by R. Thomason in [21]. Let an algebraic group G act on a variety X over a field F. We consider G-modules, i.e., $\mathcal{O}_X$-modules over X that are equipped with an G-action compatible with one on X. As in the non-equivariant case there are two categories: the abelian category $\mathcal{M}(G;X)$ of coherent G-modules and the full subcategory $\mathcal{P}(G;X)$ consisting of locally free $\mathcal{O}_X$-modules. The groups $K'_n(G;X)$ and $K_n(G;X)$ are defined as the K-groups of these two categories respectively.

In the second section we present definitions and formulate basic theorems in the equivariant K-theory such as the localization theorem, projective bundle theorem, strong homotopy invariance property and duality theorem for regular varieties.

In the following section we define an additive category $\mathcal{C}(G)$ of G-equivariant K-correspondences that was introduced by I. Panin in [15]. This category is analogous to the category of Chow correspondences presented in [9]. Many interesting functors in the equivariant K-theory of algebraic varieties factor through $\mathcal{C}(G)$. The category $\mathcal{C}(G)$ has more objects (for example, separable F-algebras are also the objects of $\mathcal{C}(G)$) and has much more morphisms than the category of G-varieties. For instance, every projective homogeneous variety is isomorphic to a separable algebra (Theorem 16).

In Sect. 2.4, we consider the equivariant K-theory of projective homogeneous varieties developed by I. Panin in [15]. The following section is devoted to the computation of the K-groups of toric models and toric varieties (see [12]).

In Sects. 2.6 and 2.7, we construct a spectral sequence

$$E^2_{p,q} = \operatorname{Tor}_p^{R(G)}\left(\mathbb{Z}, K'_q(G;X)\right) \Rightarrow K'_{p+q}(X)\ ,$$

where G is a split reductive group with the simply connected commutator subgroup and X is a G-variety.

The rest of the paper addresses the following question. Let G be an algebraic group. Under what condition on G the G-action on a G-variety X can be extended to a linear action on every vector bundle $E \to X$ making it a G-vector bundle on X? If $X = G$ and E is a line bundle, then the existence of a G-structure on E implies that E is trivial. Thus, if the answer is positive, the Picard group $\operatorname{Pic}(G)$ must be trivial. It turns out that the triviality of $\operatorname{Pic}(G)$ implies positive solution at least stably, on the level of coherent G-modules. We prove that for a factorial group G the restriction homomorphism $K'_0(G;X) \to K_0(X)$ is surjective (Theorem 39). Our exposition is different from the one presented in [11].

In the last section we consider some applications.

We use the word *variety* for a separated scheme of finite type over a field. If X is a variety over a field F and L/F is a field extension, then we write X_L for the variety $X \otimes_F L$ over L. By X_{sep} we denote $X_{F_{\mathrm{sep}}}$, where F_{sep} is a separable closure of F. If R is a commutative F-algebra, we write $X(R)$ for the set $\operatorname{Mor}_F(\operatorname{Spec} R, X)$ of R-points of X.

An *algebraic group* is a smooth affine group variety of finite type over a field.

2.2 Basic Results in the Equivariant K-theory

In this section we review the equivariant K-theory developed by R. Thomason in [21].

2.2.1 Definitions

Let G be an algebraic group over a field F. A variety X over F is called a *G-variety* if an action morphism $\theta : G \times X \to X$ of the group G on X is given, which satisfies the usual associative and unital identities for an action. In other words, to give a structure of a G-variety on a variety X is to give, for every commutative F-algebra R, a natural in R action of the group of R-points $G(R)$ on the set $X(R)$.

A *G-module M over X* is a quasi-coherent $\mathcal{O}_X$-module M together with an isomorphism of $\mathcal{O}_{G\times X}$-modules

$$\rho = \rho_M : \theta^*(M) \xrightarrow{\sim} p_2^*(M) \ ,$$

(where $p_2 : G \times X \to X$ is the projection), satisfying the cocycle condition

$$p_{23}^*(\rho) \circ (\mathrm{id}_G \times \theta)^*(\rho) = (m \times \mathrm{id}_X)^*(\rho) \ ,$$

where $p_{23} : G \times G \times X \to G \times X$ is the projection and $m : G \times G \to G$ is the product morphism (see [14, Ch. 1, §3] or [21]).

A morphism $\alpha : M \to N$ of G-modules is called a *G-morphism* if

$$\rho_N \circ \theta^*(\alpha) = p_2^*(\alpha) \circ \rho_M \ .$$

Let M be a quasi-coherent $\mathcal{O}_X$-module. For a point $x : \operatorname{Spec} R \to X$ of X over a commutative F-algebra R, write $M(x)$ for the R-module of global sections of the sheaf $x^*(M)$ over $\operatorname{Spec} R$. Thus, M defines the functor sending R to the family $\{M(x)\}$ of R-modules indexed by the R-valued point $x \in X(R)$. To give a G-module structure on M is to give natural in R isomorphisms of R-modules

$$\rho_{g,x} : M(x) \to M(gx)$$

for all $g \in G(R)$ and $x \in X(R)$ such that $\rho_{gg',x} = \rho_{g,g'x} \circ \rho_{g',x}$.

Example 1. Let X be a G-variety. A *G-vector bundle on X* is a vector bundle $E \to X$ together with a linear G-action $G \times E \to E$ compatible with the one on X. The sheaf of sections P of a G-vector bundle E has a natural structure of a G-module. Conversely, a G-module structure on the sheaf P of sections of a vector bundle $E \to X$ yields structure of a G-vector bundle on E. Indeed, for a commutative F-algebra R and a point $x \in X(R)$, the fiber of the map $E(R) \to X(R)$ over x is canonically isomorphic to $P(x)$.

We write $\mathcal{M}(G;X)$ for the abelian category of coherent G-modules over a G-variety X and G-morphisms. We set for every $n \geq 0$:

$$K'_n(G;X) = K_n\left(\mathcal{M}(G;X)\right) .$$

A flat morphism $f : X \to Y$ of varieties over F induces an exact functor

$$\mathcal{M}(G;Y) \to \mathcal{M}(G;X), \quad M \mapsto f^*(M)$$

and therefore defines the *pull-back* homomorphism

$$f^* : K'_n(G;Y) \to K'_n(G;X) .$$

A *G-projective morphism* $f : X \to Y$ is a morphism that factors equivariantly as a closed embedding into the projective bundle variety $\mathbb{P}(E)$, where E is a G-vector bundle on Y. Such a morphism f yields the *push-forward* homomorphisms [21, 1.5]

$$f_* : K'_n(G;X) \to K'_n(G;Y) .$$

If G is the trivial group, then $\mathcal{M}(G;X) = \mathcal{M}(X)$ is the category of coherent $\mathcal{O}_X$-modules over X and therefore, $K'_n(G;X) = K'_n(X)$.

Consider the full subcategory $\mathcal{P}(G;X)$ of $\mathcal{M}(G;X)$ consisting of locally free $\mathcal{O}_X$-modules. This category is naturally equivalent to the category of vector G-vector bundles on X (Example 1). The category $\mathcal{P}(G;X)$ has a natural structure of an exact category. We set

$$K_n(G;X) = K_n\left(\mathcal{P}(G;X)\right) .$$

The functor $K_n(G;*)$ is contravariant with respect to arbitrary G-morphisms of G-varieties. If G is a trivial group, we have $K_n(G;X) = K_n(X)$.

The tensor product of G-modules induces a ring structure on $K_0(G;X)$ and a module structure on $K_n(G;X)$ and $K'_n(G;X)$ over $K_0(G;X)$.

The inclusion of categories $\mathcal{P}(G;X) \hookrightarrow \mathcal{M}(G;X)$ induces an homomorphism

$$K_n(G;X) \to K'_n(G;X) .$$

Example 2. Let $\mu : G \to \mathbf{GL}(V)$ be a finite dimensional representation of an algebraic group G over a field F. One can view the G-module V as a G-vector bundle on $\operatorname{Spec} F$. Clearly, we obtain an equivalence of the abelian category $\operatorname{Rep}(G)$ of finite dimensional representations of G and the categories $\mathcal{P}(G;\operatorname{Spec} F) = \mathcal{M}(G;\operatorname{Spec} F)$. Hence there are natural isomorphisms

$$R(G) \xrightarrow{\sim} K_0(G;\operatorname{Spec} F) \xrightarrow{\sim} K'_0(G;\operatorname{Spec} F) ,$$

where $R(G) = K_0\big(\operatorname{Rep}(G)\big)$ is the *representation ring* of G. For every G-variety X over F, the pull-back map

$$R(G) \simeq K_0(G;\operatorname{Spec} F) \to K_0(G;X)$$

with respect to the structure morphism $X \to \operatorname{Spec} F$ is a ring homomorphism, making $K_0(G;X)$ (and similarly $K'_0(G;X)$) a module over $R(G)$. Note that as a group, $R(G)$ is free abelian with basis given by the classes of all irreducible representations of G over F.

Let $\pi : H \to G$ be an homomorphism of algebraic groups over F and let X be a G-variety over F. The composition

$$H \times X \overset{\pi\times \mathrm{id}_X}{\to} G \times X \overset{\theta}{\to} X$$

makes X an H-variety. Given a G-module M with the G-module structure defined by an isomorphism ϱ, we can introduce an H-module structure on M via $(\pi\times \mathrm{id}_X)^*(\varrho)$. Thus, we obtain exact functors

$$\mathrm{Res}_\pi : \mathcal{M}(G;X) \to \mathcal{M}(H;X)\ , \qquad \mathrm{Res}_\pi : \mathcal{P}(G;X) \to \mathcal{P}(H;X)$$

inducing the *restriction* homomorphisms

$$\mathrm{res}_\pi : K'_n(G;X) \to K'_n(H;X)\ , \qquad \mathrm{res}_\pi : K_n(G;X) \to K_n(H;X)\ .$$

If H is a subgroup of G, we write $\mathrm{res}_{G/H}$ for the restriction homomorphism res_π, where $\pi : H \hookrightarrow G$ is the inclusion.

2.2.2 Torsors

Let G and H be algebraic groups over F and let $f : X \to Y$ be a $G\times H$-morphism of $G \times H$-varieties. Assume that f is a G-torsor (in particular, G acts trivially on Y). Let M be a coherent H-module over Y. Then $f^*(M)$ has a structure of a coherent $G\times H$-module over X given by $p^*(\varrho_M)$, where p is the composition of the projection $G \times H \times X \to H \times X$ and the morphism $\mathrm{id}_H \times f : H \times X \to H \times Y$.

Thus, there are exact functors

$$f^0 : \mathcal{M}(H;Y) \to \mathcal{M}(G \times H;X)\ , \qquad M \mapsto p^*(M)\ ,$$

$$f^0 : \mathcal{P}(H;Y) \to \mathcal{P}(G \times H;X)\ , \qquad P \mapsto p^*(P)\ .$$

3 **Proposition 3** (Cf. [21, Prop. 6.2]) The functors f^0 are equivalences of categories. In particular, the homomorphisms

$$K'_n(H;Y) \to K'_n(G \times H;X)\ ,$$

$$K_n(H;Y) \to K_n(G \times H;X)\ ,$$

induced by f^0, are isomorphisms.

Proof Under the isomorphisms

$$G \times X \tilde{\to} X \times_Y X , \qquad (g,x) \mapsto (gx,x) ,$$

$$G \times G \times X \tilde{\to} X \times_Y X \times_Y X , \qquad (g,g',x) \mapsto (gg'x, g'x, x)$$

the action morphism θ is identified with the first projection $p_1 : X \times_Y X \to X$ and the morphisms $m \times \mathrm{id}$, $\mathrm{id} \times \theta$ are identified with the projections $p_{13}, p_{12} : X \times_Y X \times_Y X \to X \times_Y X$. Hence, the isomorphism ρ giving a G-module structure on a $\mathcal{O}_X$-module M can be identified with the *descent data*, i.e. with an isomorphism

$$\varphi : p_1^*(M) \tilde{\to} p_2^*(M)$$

of $O_{X \times_Y X}$-modules satisfying the usual cocycle condition

$$p_{23}^*(\varphi) \circ p_{12}^*(\varphi) = p_{13}^*(\varphi) .$$

More generally, a $G \times H$-module structure on M is the descent data commuting with an H-module structure on M. The statement follows now from the theory of faithfully flat descent [13, Prop.2.22].

Example 4. Let $f : X \to Y$ be a G-torsor and let $\rho : G \to \mathbf{GL}(V)$ be a finite dimensional representation. The group G acts linearly on the affine space $\mathbb{A}(V)$ of V, so that the product $X \times \mathbb{A}(V)$ is a G-vector bundle on X. We write E_ρ for the vector bundle on Y such that $f^*(E_\rho) \simeq X \times \mathbb{A}(V)$, i.e., $E_\rho = G\backslash\big(X \times \mathbb{A}(V)\big)$. The assignment $\rho \mapsto E_\rho$ gives rise to a group homomorphism

$$r : R(G) \to K_0(Y) .$$

Note that the homomorphism r coincides with the composition

$$R(G) \tilde{\to} K_0(G; \operatorname{Spec} F) \xrightarrow{p^*} K_0(G; X) \tilde{\to} K_0(Y) ,$$

where $p : X \to \operatorname{Spec} F$ is the structure morphism.

Let G be an algebraic group over F and let H be a subgroup of G.

Corollary 5 For every G-variety X, there are natural isomorphisms 5

$$K_n\big(G; X \times (G/H)\big) \simeq K_n(H; X) , \quad K_n'\big(G; X \times (G/H)\big) \simeq K_n'(H; X) .$$

Proof Consider $X \times G$ as a $G \times H$-variety with the action morphism given by the rule $(g,h) \cdot (x, g') = (hx, gg'h^{-1})$. The statement follows from Proposition 3 applied to the G-torsor $p_2 : X \times G \to X$ and to the H-torsor $X \times G \to X \times (G/H)$ given by $(x, g) \mapsto (gx, gH)$.

Let $\varrho : H \to \mathbf{GL}(V)$ be a finite dimensional representation. Consider G as an H-torsor over G/H with respect to the H-action given by $h * g = gh^{-1}$. The vector bundle $E_\varrho = H\backslash(G \times \mathbb{A}(V))$ constructed in Example 4 has a natural structure of a G-vector bundle. Corollary5 with $X = \operatorname{Spec} F$ implies:

6 **Corollary 6** The assignment $\varrho \mapsto E_\varrho$ gives rise to an isomorphism $R(H) \xrightarrow{\sim} K_0(G; G/H)$.

7 **Corollary 7** There is a natural isomorphism $K_n(G/H) \xrightarrow{\sim} K_n(H; G)$.

Proof Apply Proposition 3 to the H-torsor $G \to G/H$.

2.2.3 Basic Results in Equivariant K-theory

We formulate basic statements in the equivariant algebraic K-theory developed by R. Thomason in [21]. In all of them G is an algebraic group over a field F and X is a G-variety.

Let $Z \subset X$ be a closed G-subvariety and let $U = X \setminus Z$. Since every coherent G-module over U extends to a coherent G-module over X [21, Cor. 2.4], the category $\mathcal{M}(G; U)$ is equivalent to the factor category of $\mathcal{M}(G; X)$ by the subcategory $\mathcal{M}'$ of coherent G-modules supported on Z. By Quillen's devissage theorem [17, §5, Th. 4], the inclusion of categories $\mathcal{M}(G; Z) \subset \mathcal{M}'$ induces an isomorphism $K'_n(G; Z) \xrightarrow{\sim} K'_n(\mathcal{M}')$. The localization in algebraic K-theory [17, §5, Th. 5] yields *connecting homomorphisms*

$$K'_{n+1}(G; U) \xrightarrow{\delta} K'_n(\mathcal{M}') \simeq K'_n(G; Z)$$

and the following:

8 **Theorem 8** [21, Th. 2.7] (Localization) The sequence

$$\ldots \to K'_{n+1}(G; U) \xrightarrow{\delta} K'_n(G; Z) \xrightarrow{i_*} K'_n(G; X) \xrightarrow{j^*} K'_n(G; U) \xrightarrow{\delta} \ldots ,$$

where $i : Z \to X$ and $j : U \to X$ are the embeddings, is exact.

9 **Corollary 9** Let X be a G-variety. Then the natural closed G-embedding $f : X_{\mathrm{red}} \to X$ induces the isomorphism $f_* : K_n(G; X_{\mathrm{red}}) \to K_n(G; X)$.

Let X be a G-variety and let E be a G-vector bundle of rank $r + 1$ on X. The projective bundle variety $\mathbb{P}(E)$ has natural structure of a G-variety so that the natural morphism $p : \mathbb{P}(E) \to X$ is G-equivariant. We write $\mathcal{L}$ for the G-module of sections of the tautological line bundle on $\mathbb{P}(E)$.

A modification of the Quillen's proof [17, §8] of the standard projective bundle theorem yields:

Theorem 10 [21, Th. 3.1] (Projective bundle theorem) The correspondence 10

$$(a_0, a_1, \dots, a_r) \mapsto \sum_{i=0}^{r} \left[\mathcal{L}^{\otimes i}\right] \otimes p^* a_i$$

induces isomorphisms

$$K_n(G; X)^{r+1} \to K_n\big(G; \mathbb{P}(E)\big) , \quad K'_n(G; X)^{r+1} \to K'_n\big(G; \mathbb{P}(E)\big) .$$

Let X be a G-variety and let $E \to X$ be a G-vector bundle on X. Let $f : Y \to X$ be a torsor under the vector bundle variety E (considered as a group scheme over X) and G acts on Y so that f and the action morphism $E \times_X Y \to Y$ are G-equivariant. For example, one can take the trivial torsor $Y = E$.

Theorem 11 [21, Th. 4.1] (Strong homotopy invariance property) The pull-back homomorphism 11

$$f^* : K'_n(G; X) \to K'_n(G; Y)$$

is an isomorphism.

The idea of the proof is construct an exact sequence of G-vector bundles on X:

$$0 \to E \to W \xrightarrow{\varphi} \mathbb{A}^1_X \to 0 ,$$

where $\mathbb{A}^1_X$ is the trivial line bundle, such that $\varphi^{-1}(1) \simeq Y$. Thus, Y is isomorphic to the open complement of the projective bundle variety $\mathbb{P}(E)$ in $\mathbb{P}(V)$. Then one uses the projective bundle theorem and the localization to compute the equivariant K'-groups of Y.

Corollary 12 Let $G \to \mathbf{GL}(V)$ be a finite dimensional representation. Then the projection $p : X \times \mathbb{A}(V) \to X$ induces the pull-back isomorphism 12

$$p^* : K'_n(G; X) \xrightarrow{\sim} K'_n\big(G; X \times \mathbb{A}(V)\big) .$$

Let X be a regular G-variety. By [21, Lemma 5.6], every coherent G-module over X is a factor module of a locally free coherent G-module. Therefore, every coherent G-module has a finite resolution by locally free coherent G-modules. The resolution theorem [17, §4, Th. 3] then yields:

13 **Theorem 13** [21, Th. 5.7] (Duality for regular varieties) Let X be a regular G-variety over F. Then the canonical homomorphism $K_n(G;X) \to K'_n(G;X)$ is an isomorphism.

2.3 Category $\mathcal{C}(G)$ of G-equivariant K-correspondences

Let G be an algebraic group over a field F and let A be a separable F-algebra, i.e. A is isomorphic to a product of simple algebras with centers separable field extensions of F. An *G-A-module* over a G-variety X is a G-module M over X which is endowed with the structure of a left $A \otimes_F \mathcal{O}_X$-module such that the G-action on M is A-linear. An *G-A-morphism* of G-A-modules is a G-morphism that is also a morphism of $A \otimes_F \mathcal{O}_X$-modules.

We consider the abelian category $\mathcal{M}(G;X,A)$ of G-A-modules and G-A-morphisms and set

$$K'_n(G;X,A) = K_n(\mathcal{M}(G;X,A)) \ .$$

The functor $K'_n(G;*,A)$ is contravariant with respect to flat G-A-morphisms and is covariant with respect to projective G-A-morphisms of G-varieties. The category $\mathcal{M}(G;X,F)$ is isomorphic to $\mathcal{M}(G;X)$, and thus it follows that $K'_n(G;X,F) = K'_n(G;X)$.

Consider also the full subcategory $\mathcal{P}(G;X,A)$ of $\mathcal{M}(G;X,A)$ consisting of all G-A-modules which are locally free $\mathcal{O}_X$-modules. The K-groups of the category $\mathcal{P}(G;X,A)$ are denoted by $K_n(G;X,A)$. The group $K_n(G;X,F)$ coincides with $K_n(G;X)$.

In [15], I. Panin has defined the *category of G-equivariant K-correspondences* $\mathcal{C}(G)$ whose objects are the pairs (X,A), where X is a smooth projective G-variety over F and A is a separable F-algebra. Morphisms in $\mathcal{C}(G)$ are defined as follows:

$$\mathrm{Mor}_{\mathcal{C}(G)}((X,A),(Y,B)) = K_0(G;X \times Y, A^{\mathrm{op}} \otimes_F B) \ ,$$

where A^{op} stands for the algebra opposite to A. If $u : (X,A) \to (Y,B)$ and $v : (Y,B) \to (Z,C)$ are two morphisms in $\mathcal{C}(G)$, then their composition is defined by the formula

$$v \circ u = p_{13*}(p_{23}^*(v) \otimes_B p_{12}^*(u)) \ ,$$

where p_{12}, p_{13} and p_{23} are the projections from $X \times Y \times Z$ to $X \times Y$, $X \times Z$ and $Y \times Z$ respectively. The identity endomorphism of (X,A) in $\mathcal{C}(G)$ is the class $[A \otimes_F \mathcal{O}_\Delta]$, where $\Delta \subset X \times X$ is the diagonal, in the group

$$K'_0(G;X \times X, A^{\mathrm{op}} \otimes_F A) \simeq K_0(G;X \times X, A^{\mathrm{op}} \otimes_F A) = \mathrm{End}_{\mathcal{C}(G)}(X,A) \ .$$

We will simply write X for (X, F) and A for $(\operatorname{Spec} F, A)$ in $\mathcal{C}(G)$.

The category $\mathcal{C}(G)$ for the trivial group G is simply denoted by $\mathcal{C}$. There is the forgetful functor $\mathcal{C}(G) \to \mathcal{C}$.

Note that an element $u \in K_0(G; X \times Y, A^{\mathrm{op}} \otimes_F B)$, i.e. a morphism $u : (X, A) \to (Y, B)$ can be considered also as a morphism $u^{\mathrm{op}} : (Y, B^{\mathrm{op}}) \to (X, A^{\mathrm{op}})$. Thus, the category $\mathcal{C}(G)$ has the *involution functor* taking (X, A) to (X, A^{op}).

For every variety Z over F and every $n \in \mathbb{Z}$ we have the *realization functor*

$$\mathcal{K}_n^Z : \mathcal{C}(G) \to \textbf{Abelian Groups} ,$$

taking a pair (X, A) to $K_n'(G; Z \times X, A)$ and a morphism

$$v \in \operatorname{Hom}_{\mathcal{C}(G)} \big((X, A), (Y, B)\big) = K_0(G; X \times Y, A^{\mathrm{op}} \otimes_F B)$$

to

$$\mathcal{K}_n^Z(v) : K_n'(G; Z \times X, A) \to K_n'(G; Z \times Y, B)$$

given by the formula

$$\mathcal{K}_n^Z(v)(u) = v \circ u .$$

Note that we don't need to assume Z neither smooth nor projective to define $\mathcal{K}_n^Z$. We simply write $\mathcal{K}_n$ for $\mathcal{K}_n^{\operatorname{Spec} F}$.

Example 14 . Let X be a smooth projective variety over F. The identity $[\mathcal{O}_X] \in K_0(X)$ defines two morphisms $u : X \to \operatorname{Spec} F$ and $v : \operatorname{Spec} F \to X$ in $\mathcal{C}$. If $p_*[\mathcal{O}_X] = 1 \in K_0(F)$, where $p : X \to \operatorname{Spec} F$ is the structure morphism (for example, if X is a projective homogeneous variety), then the composition $u \circ v$ in $\mathcal{C}$ is the identity. In other words, the morphism p splits canonically in $\mathcal{C}$, i.e., the point $\operatorname{Spec} F$ is a canonical "direct summand" of X in $\mathcal{C}$, although X may have no rational points. The application of the resolution functor $\mathcal{K}_n^Z$ for a variety Z over F shows that the group $K_n'(Z)$ is a canonical direct summand of $K_n'(X \times Z)$.

Let G be a split reductive group over a field F with simply connected commutator subgroup and let $B \subset G$ be a Borel subgroup. By [20, Th.1.3], $R(B)$ is a free $R(G)$-module.

The following statement is a slight generalization of [15, Th. 6.6].

Proposition 15 Let $Y = G/B$ and let $u_1, u_2, \ldots, u_m$ be a basis of $R(B) = K_0(G; Y)$ over $R(G)$. Then the element 15

$$u = (u_i) \in R(B)^m = K_0(G; Y)^m = K_0(G; Y, F^m)$$

defines an isomorphism $F^m \xrightarrow{\sim} Y$ in the category $\mathcal{C}(G)$.

Proof Denote by $p : G/B \to \operatorname{Spec} F$ the structure morphism. Since G/B is a projective variety, the push-forward homomorphism

$$p_* : R(B) = K_0(G; G/B) \to K_0(G; \operatorname{Spec} F) = R(G)$$

is well defined. The $R(G)$-bilinear form on $R(B)$ defined by the formula

$$\langle u, v\rangle_G = p_*(u \cdot v)$$

is unimodular ([6], [15, Th. 8.1.], [11, Prop. 2.17]).

Let $v_1, v_2, ..., v_m$ be the dual $R(G)$-basis of $R(B)$ with respect to the unimodular bilinear form. The element $v = (v_i) \in K_0(G; Y, F^m)$ can be considered as a morphism $Y \to F^m$ in $\mathcal{C}(G)$. The fact that u and v are dual bases is equivalent to the equality $v \circ u = \mathrm{id}$. In order to prove that $u \circ v = \mathrm{id}$ it suffices to show that the $R(G)$-module $K_0(G, Y \times Y)$ is generated by m^2 elements (see [15, Cor. 7.3]). It is proved in [15, Prop. 8.4] for a simply connected group G, but the proof goes through for a reductive group G with simply connected commutator subgroup.

2.4 Equivariant K-theory of Projective Homogeneous Varieties

Let G be a semisimple group over a field F. A G-variety X is called *homogeneous* (resp. *projective homogeneous*) if X_{sep} is isomorphic (as a G_{sep}-variety) to G_{sep}/H for a closed (resp. a (reduced) parabolic) subgroup $H \subset G_{\mathrm{sep}}$.

2.4.1 Split Case

Let G be a simply connected split algebraic group over F, let $P \subset G$ be a parabolic subgroup and set $X = G/P$. The center C of G is a finite diagonalizable group scheme and $C \subset P$; we write C^* for the character group of C. For a character $\chi \in C^*$, we say that a representation $\rho : P \to \mathbf{GL}(V)$ is *χ-homogeneous* if the restriction of ρ on C is given by multiplication by χ. Let $R(P)^{(\chi)}$ be the subgroup of $R(P)$ generated by the classes of χ-homogeneous representations of P.

By [20, Th.1.3], there is a basis $u_1, u_2, ..., u_k$ of $R(P)$ over $R(G)$ such that each $u_i \in R(P)^{(\chi_i)}$ for some $\chi_i \in C^*$. As in the proof of Proposition 15, the elements u_i define an isomorphism $u : E \to X$ in the category $\mathcal{C}(G)$, where $E = F^k$.

For every $i = 1, 2, ..., k$, choose a representation $\rho_i : G \to \mathbf{GL}(V_i)$ such that $[\rho_i] \in R(G)^{(\chi_i)}$. Consider the vector spaces V_i as G-vector bundles on $\operatorname{Spec} F$ with trivial G-action. The classes of the dual vector spaces

$$v_i = [V_i^*] \in K_0\big(G; \operatorname{Spec} F, \operatorname{End}(V_i^*)\big)$$

define isomorphisms $v_i : \mathrm{End}(V_i) \to F$ in $\mathcal{C}(G)$. Let V be the E-module $V_1 \times V_2 \times \cdots \times V_k$. Taking the product of all the v_i we get an isomorphism $v : \mathrm{End}_E(V) \to E$ in $\mathcal{C}(G)$. The composition $w = u \circ v$ is then an isomorphism $w : \mathrm{End}_E(V) \to X$.

Now we let the group G act on itself by conjugation, on X by left translations, on w via the representations ϱ_i. Let $\overline{G} = G/C$ be the adjoint group associated with G. We claim that all the G-actions factor through $\overline{G}$. This is obvious for the actions on G and X. Since the elements u_i are χ_i-homogeneous and the center C acts on V_i^* via ϱ_i by the character χ_i^{-1}, the class w also admits a $\overline{G}$-structure.

Quasi-split Case 2.4.2

Let G be a simply connected quasi-split algebraic group over F, let $P \subset G$ be a parabolic subgroup and set $X = G/P$. The absolute Galois group $\Gamma = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ acts naturally on the representation ring $R(P_{\mathrm{sep}})$. By [20, Th.1.3], the basis $u_1, u_2, \ldots, u_k \in R(P_{\mathrm{sep}})$ over $R(G_{\mathrm{sep}})$ considered in 2.4.1, can be chosen Γ-invariant. Let E be the étale F-algebra corresponding to the Γ-set of the u_i's. As in the proof of Proposition 15, the element $u \in K_0(G; X, E)$ defines an isomorphism $u : E \to X$ in the category $\mathcal{C}(G)$.

Since the group Γ permutes the χ_i defined in 2.4.1, one can choose the representations ϱ_i whose classes in the representation ring $R(G_{\mathrm{sep}})$ are also permuted by Γ. Hence as in 2.4.1, there is an E-module V and an isomorphism $w : \mathrm{End}_E(V) \to X$ which admits a $\overline{G}$-structure.

General Case 2.4.3

Let G be a simply connected algebraic group over F, let X be a projective homogeneous variety of G. Choose a quasi-split inner twisted form G^q of G. The group G is obtained from G^q by twisting with respect to a cocycle γ with coefficients in the quasi-split adjoint group $\overline{G}^q$. Let X^q be the projective homogeneous G^q-variety which is a twisted form of X. As in 2.4.2, find an isomorphism $w^q : \mathrm{End}_E(V) \to X^q$ in $\mathcal{C}(G^q)$ for a certain étale F-algebra E and an E-module V. Note that all the structures admit $\overline{G}^q$-operators. Twisting by the cocycle γ we get an isomorphism $w : A \to X$ in $\mathcal{C}(G)$ for a separable F-algebra A with center E. We have proved

Theorem 16 (Cf. [15, Th. 12.4]) Let G be a simply connected group over a field F and let X be a projective homogeneous G-variety. Then there exist a separable F-algebra A and an isomorphism $A \simeq X$ in the category $\mathcal{C}(G)$. In particular, $K_*(G; X) \simeq K_*(G; A)$ and $K_*(X) \simeq K_*(A)$. 16

Corollary 17 The restriction homomorphism $K_0(G; X) \to K_0(X)$ is surjective. 17

Proof The statement follows from the surjectivity of the restriction homomorphism $K_0(G; A) \to K_0(A)$.

We will generalize Corollary 17 in Theorem 39.

2.5 K-theory of Toric Varieties

Let a torus T act on a normal geometrically irreducible variety X defined over a field F. The variety X is called a *toric T-variety* if there is an open orbit which is a principal homogeneous space of T. A toric T-variety is called a *toric model of T* if the open orbit has a rational point. A choice of a rational point x in the open orbit gives an open T-equivariant embedding $T \hookrightarrow X$, $t \mapsto tx$.

2.5.1 K-theory of Toric Models

We will need the following:

18 **Proposition 18** [12, Proposition 5.6] Let X be a smooth toric T-model defined over a field F. Then there is a torus S over F, an S-torsor $\pi : U \to X$ and an S-equivariant open embedding of U into an affine space $\mathbb{A}$ over F on which S acts linearly.

19 **Remark 19** It turns out that the canonical homomorphism $S^*_{\mathrm{sep}} \to \mathrm{Pic}(X_{\mathrm{sep}})$ is an isomorphism, so that $\pi : U \to X$ is the *universal torsor* in the sense of [2, 2.4.4]. Thus, the Proposition 18 asserts that the universal torsor of X can be equivariantly imbedded into an affine space as an open subvariety.

Let $\varrho : S \to \mathbf{GL}(V)$ be a representation over F. Suppose that there is an action of an étale F-algebra A on V commuting with the S-action. Then A acts on the vector bundle E_ϱ (see Example 4), therefore, E_ϱ defines an element $u_\varrho \in K_0(X, A)$, i.e., a morphism $u_\varrho : A \to X$ in $\mathcal{C}$. The composition

$$K_0(A) \xrightarrow{\alpha_\varrho} R(S) \xrightarrow{r} K_0(X) ,$$

where r is defined in Example 4 and α_ϱ is induced by the exact functor $M \mapsto M \otimes_A V$, is given by the rule $x \mapsto u_\varrho \circ x$.

Let ϱ be an irreducible representation. Since S is a torus, ϱ is the corestriction in a finite separable field extension L_ϱ/F of a 1-dimensional representation of S. Thus, there is an action of L_ϱ on V that commutes with the S-action. Note that the element u_ϱ defined above is represented by an element of the Picard group $\mathrm{Pic}(X \otimes_F L_\varrho)$.

Now we consider two irreducible representations ϱ and μ of the torus S over F, and apply the construction described above to the torus $S \times S$ and its representation

$$\varrho \otimes \mu : S \times S \to \mathbf{GL}(V_\varrho \otimes_F V_\mu) \ .$$

The composition

$$K_0(L_\varrho \otimes_F L_\mu) \xrightarrow{\alpha_{\varrho,\mu}} R(S \times S) \xrightarrow{r} K_0(X \times X)$$

coincides with the map

$$x \mapsto u_\varrho^{\mathrm{op}} \circ x \circ u_\mu \ ,$$

where the composition is taken in $\mathcal{C}$ and $u_\mu : X \to L_\mu$, $u_\varrho^{\mathrm{op}} : L_\varrho \to X$, $x : L_\mu \to L_\varrho$ are considered as the morphisms in $\mathcal{C}$.

Now let Φ be a finite set of irreducible representations of S. Set

$$A = \prod_{\varrho \in \Phi} L_\varrho, \quad u = \sum_{\varrho \in \Phi} u_\varrho, \quad \alpha = \sum_{\varrho,\mu \in \Phi} \alpha_{\varrho,\mu} \ .$$

The element u_ϱ is represented by an element of the Picard group $\mathrm{Pic}(X \otimes_F A)$.

Then the composition

$$K_0(A \otimes_F A) \xrightarrow{\alpha} R(S \times S) \xrightarrow{r} K_0(X \times X)$$

is given by the rule $x \mapsto u^{\mathrm{op}} \circ x \circ u$, where u is considered as a morphism $X \to A$.

The homomorphism r coincides with the composition

$$R(S \times S) = K_0(S \times S; \mathrm{Spec}\, F) \xrightarrow{\sim} K_0(S \times S; \mathbb{A} \times \mathbb{A}) \twoheadrightarrow$$

$$K_0(S \times S; U \times U) = K_0(X \times X)$$

and hence r is surjective. By the representation theory of algebraic tori, the sum of all the $\alpha_{\varrho,\mu}$ is an isomorphism. It follows that for sufficiently large (but finite!) set Φ of irreducible representations of S the identity $\mathrm{id}_X \in K_0(X \times X)$ belongs to the image of $r \circ \alpha$. In other words, there exists $x \in K_0(A \otimes_F A)$ such that $u^{\mathrm{op}} \circ x \circ u = \mathrm{id}_X$ in $\mathcal{C}$, i.e. $v = u^{\mathrm{op}} \circ x$ is a left inverse to $u : X \to A$ in $\mathcal{C}$. We have proved the following:

Theorem 20 [12, Th. 5.7] Let X be a smooth projective toric model of an algebraic torus defined over a field F. Then there exist an étale F-algebra A and elements u, 20
$v \in K_0(X, A)$ such that the composition $X \xrightarrow{u} A \xrightarrow{v} X$ in $\mathcal{C}$ is the identity and u is represented by a class in $\mathrm{Pic}(X \otimes_F A)$.

K-theory of Toric Varieties 2.5.2

Let T be a torus over F. The natural G-equivariant bilinear map

$$T(F_{\mathrm{sep}}) \otimes T^*_{\mathrm{sep}} \to F^\times_{\mathrm{sep}} \ , \qquad x \otimes \chi \mapsto \chi(x)$$

induces a pairing of the Galois cohomology groups

$$H^1\big(F, T(F_{sep})\big) \otimes H^1(F, T^*_{sep}) \to H^2(F, F^\times_{sep}) = \mathrm{Br}(F) ,$$

where $\mathrm{Br}(F)$ is the Brauer group of F. There is a natural isomorphism $\mathrm{Pic}(T) \simeq H^1(F, T^*_{sep})$ (see [23]). A principal homogeneous T-space U defines an element $[U] \in H^1\big(F, T(F_{sep})\big)$. Therefore, the pairing induces the homomorphism

$$\lambda^U : \mathrm{Pic}(T) \to \mathrm{Br}(F) , \qquad [Q] \mapsto [U] \cup [Q] .$$

Let X be a toric variety of the torus T with the open orbit U which is a principal homogeneous space over T.

21 **Theorem 21** [12, Th. 7.6] Let Y be a smooth projective toric variety over a field F. Then there exist an étale F-algebra A, a separable F-algebra B of rank n^2 over its center A and morphisms $u : Y \to B$, $v : B \to Y$ in $\mathcal{C}$ such that $v \circ u = \mathrm{id}$. The morphism u is represented by a locally free $\mathcal{O}_Y$-module in $\mathcal{P}(Y, B)$ of rank n. The class of the algebra B in $\mathrm{Br}(A)$ belongs to the image of $\lambda^{U_A} : \mathrm{Pic}(T_A) \to \mathrm{Br}(A)$.

22 **Corollary 22** The homomorphism $\mathcal{K}_n(u) : K_n(X) \to K_n(A)$ identifies $K_n(X)$ with the direct summand of $K_n(A)$ which is equal to the image of the projector $\mathcal{K}_n(u \circ v) : K_n(A) \to K_n(A)$. In particular, $K_0(X)$ is a free abelian group of finite rank.

2.6 Equivariant K-theory of Solvable Algebraic Groups

We consider separately the equivariant K-theory of unipotent groups and algebraic tori.

2.6.1 Split Unipotent Groups

A unipotent group U is called *split* if there is a chain of subgroups of U with the subsequent factor groups isomorphic to the additive group $\boldsymbol{G}_a$. For example, the unipotent radical of a Borel subgroup of a (quasi-split) reductive group is split.

23 **Theorem 23** Let U be a split unipotent group and let X be a U-variety. Then the restriction homomorphism $K'_n(U; X) \to K'_n(X)$ is an isomorphism.

Proof Since U is split, it is sufficient to prove that for a subgroup $U' \subset U$ with $U/U' \simeq \mathbf{G}_a$, the restriction homomorphism $K'_n(U;X) \to K'_n(U';X)$ is an isomorphism. By Corollary 5, this homomorphism coincides with the pull-back $K'_n(U;X) \to K'_n(U;X \times \mathbf{G}_a)$ with respect to the projection $X \times \mathbf{G}_a \to X$, that is an isomorphism by the homotopy invariance property (Corollary 12).

Split Algebraic Tori 2.6.2

Let T be a split torus over a field F. Choose a basis $\chi_1, \chi_2, ..., \chi_r$ of the character group T^*. We define an action of T on the affine space $\mathbb{A}^r$ by the rule $t \cdot x = y$ where $y_i = \chi_i(t)x_i$. Write H_i $(i = 1, 2, ..., r)$ for the coordinate hyperplane in $\mathbb{A}^r$ defined by the equation $x_i = 0$. Clearly, H_i is a closed T-subvariety in $\mathbb{A}^r$ and $T = \mathbb{A}^r - \cup_{i=1}^r H_i$. For every subset $I \subset \{1, 2, ..., r\}$ set $H_I = \cap_{i \in I} H_i$.

In [8], M. Levine has constructed a spectral sequence associated to a family of closed subvarieties of a given variety. This sequence generalizes the localization exact sequence. We adapt this sequence to the equivariant algebraic K-theory and also change the indices making this spectral sequence of homological type.

Let X be a T-variety over F. The family of closed subsets $Z_i = X \times H_i$ in $X \times \mathbb{A}^r$ gives then a spectral sequence

$$E^1_{p,q} = \coprod_{|I|=p} K'_q(T; X \times H_I) \Rightarrow K'_{p+q}(T; X \times T) .$$

By Corollary 5, the group $K'_{p+q}(T; X \times T)$ is isomorphic to $K'_{p+q}(X)$.

In order to compute $E^1_{p,q}$, note that H_I is an affine space over F, hence the pull-back $K'_q(T;X) \to K'_q(T; X \times H_I)$ is an isomorphism by the homotopy invariance property (Corollary 12). Thus,

$$E^1_{p,q} = \coprod_{|I|=p} K'_q(T;X) \cdot e_I$$

and by [8, p.419], the differential map $d : E^1_{p+1,q} \to E^1_{p,q}$ is given by the formula

$$d(x \cdot e_I) = \sum_{k=0}^{p} (-1)^k (1 - \chi_{i_k}^{-1}) x \cdot e_{I - \{i_k\}} , \tag{2.1}$$

where $I = \{i_0 < i_1 < \cdots < i_p\}$.

Consider the Kozsul complex C_* built upon the free $R(T)$-module $R(T)^r$ and the system of the elements $1 - \chi_i^{-1} \in R(T)$. More precisely,

$$C_p = \coprod_{|I|=p} R(T) \cdot e_I$$

and the differential $d : C_{p+1} \to C_p$ is given by the rule formally coinciding with (2.1), where $x \in R(T)$.

The representation ring $R(T)$ is the group ring over the character group T^*. The Kozsul complex gives the resolution $C_* \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ by free $R(T)$-modules, where we view $\mathbb{Z}$ as a $R(T)$-module via the *rank* homomorphism $R(T) \to \mathbb{Z}$ taking the class of a representation to its dimension. It follows from (2.1) that the complex $E^1_{*,q}$ coincides with

$$C_* \otimes_{R(T)} K'_q(T;X) .$$

Hence, being the homology group of $E^1_{*,q}$, the term $E^2_{p,q}$ is equal to

$$\mathrm{Tor}_p^{R(T)}\big(\mathbb{Z}, K'_q(T;X)\big) .$$

We have proved:

24 **Proposition 24** Let T be a split torus over a field F and let X be a T-variety. Then there is a spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{R(T)}\big(\mathbb{Z}, K'_q(T;X)\big) \Rightarrow K'_{p+q}(X) .$$

We are going to prove that if X is smooth projective, the spectral sequence degenerates.

Let G be an algebraic group and let $H \subset G$ be a subgroup. Suppose that there exists a group homomorphism $\pi : G \to H$ such that $\pi|_H = \mathrm{id}_H$. For a smooth projective G-variety X we write $\dot{X}$ for the variety X together with the new G-action $g * x = \pi(g)x$.

25 **Lemma 25** If the restriction homomorphism $K_0(G;\dot{X} \times X) \to K_0(H;X \times X)$ is surjective, then the restriction homomorphism $K'_n(G;X) \to K'_n(H;X)$ is a split surjection.

Proof Since the restriction map

$$\mathrm{res}_{G/H} : \mathrm{Hom}_{\mathcal{C}(G)}(\dot{X}, X) = K_0(G;\dot{X} \times X) \to$$

$$K_0(H;X \times X) = \mathrm{Hom}_{\mathcal{C}(H)}(X, X)$$

is surjective, there is $\nu \in \mathrm{Hom}_{\mathcal{C}(G)}(\dot{X}, X)$ such that $\mathrm{res}_{G/H}(\nu) = \mathrm{id}_X$ in $\mathcal{C}(G)$.

Consider the diagram

$$\begin{array}{ccccc} K'_n(H;X) & \overset{\mathrm{res}_\pi}{\to} & K'_n(G;\dot{X}) & \overset{\mathcal{K}_n(\nu)}{\to} & K'_n(G;X) \\ & & {\scriptstyle \mathrm{res}_{G/H}}\downarrow & & {\scriptstyle \mathrm{res}_{G/H}}\downarrow \\ & & K'_n(H;X) & = & K'_n(H;X) \end{array} ,$$

where the square is commutative since $\mathrm{res}_{G/H}(\nu) = \mathrm{id}_X$. The equality $\mathrm{res}_{G/H} \circ \mathrm{res}_\pi = \mathrm{id}$ implies that the composition in the top row splits the restriction homomorphism $K'_n(G;X) \to K'_n(H;X)$.

Let T be a split torus over F, and let $\chi \in T^*$ be a character such that $T^*/(\mathbb{Z} \cdot \chi)$ is a torsion-free group. Then $T' = \ker(\chi)$ is a subtorus in T. Denote by $\pi : T \to T'$ a splitting of the embedding $T' \hookrightarrow T$.

Proposition 26 Let X be a smooth projective T-variety. Then the restriction homomorphism $K'_n(T;X) \to K'_n(T';X)$ is a split surjection. 26

Proof We use the notation $\dot{X}$ as above. Since $T/T' \simeq \mathbf{G}_m$, by Corollary 12, Corollary 5 and the localization (Theorem 8), we have the surjection

$$K'_0(T;\dot{X} \times X) \xrightarrow{\sim} K'_0(T;\dot{X} \times X \times \mathbb{A}^1_F) \twoheadrightarrow K'_0(T;\dot{X} \times X \times \mathbf{G}_m) \simeq K'_0(T';X \times X)$$

which is nothing but the restriction homomorphism. The statement follows from Lemma 25.

Corollary 27 The sequence 27

$$0 \to K'_n(T;X) \xrightarrow{1-\chi} K'_n(T;X) \xrightarrow{\text{res}} K'_n(T';X) \to 0$$

is split exact.

Proof We consider $X \times \mathbb{A}^1_F$ as a T-variety with respect to the T-action on $\mathbb{A}^1_F$ given by the character χ. In the localization exact sequence

$$\dots \to K'_n(T;X) \xrightarrow{i_*} K'_n(T;X \times \mathbb{A}^1_F) \xrightarrow{j^*} K'_n(T;X \times \mathbf{G}_m) \xrightarrow{\delta} \dots ,$$

where $i : X = X \times \{0\} \hookrightarrow X \times \mathbb{A}^1_F$ and $j : X \times \mathbf{G}_m \hookrightarrow X \times \mathbb{A}^1_F$ are the embeddings, the second term is identified with $K'_n(T;X)$ by Corollary 12 and the third one with $K'_n(T';X)$ since $T/T' \simeq \mathbf{G}_m$ as T-varieties (Corollary 5). With these identifications, j^* is the restriction homomorphism which is a split surjection by Proposition 26. By the projection formula, i_* is the multiplication by $i_*(1)$. Let t be the coordinate of $\mathbb{A}^1$. It follows from the exactness of the sequence of T-modules over $X \times \mathbb{A}^1_F$:

$$0 \to \mathcal{O}_{X \times \mathbb{A}^1}[\chi^{-1}] \xrightarrow{t} \mathcal{O}_{X \times \mathbb{A}^1} \to i_*(\mathcal{O}_X) \to 0$$

that $i_*(1) = 1 - \chi^{-1}$.

Proposition 28 Let T be a split torus and let X be a smooth projective T-variety. Then the spectral sequence in Proposition 24 degenerates, i.e., 28

$$\mathrm{Tor}_p^{R(T)}\big(\mathbb{Z}, K'_n(T;X)\big) = \begin{cases} K'_n(X) , & \text{if } p = 0 , \\ 0 , & \text{if } p > 0 . \end{cases}$$

Proof Let $\chi_1, \chi_2 \dots, \chi_r$ be a $\mathbb{Z}$-basis of the character group T^*. Since $R(T)$ is a Laurent polynomial ring in the variables χ_i, and by Corollary 27, the elements $1-\chi_i \in R(T)$ form a $R(T)$-regular sequence, the result follows from [19, IV-7].

2.6.3 Quasi-trivial Algebraic Tori

An algebraic torus T over a field F is called *quasi-trivial* if the character Galois module T^*_{sep} is permutation. In other words, T is isomorphic to the torus $\mathbf{GL}_1(C)$ of invertible elements of an étale F-algebra C. The torus $T = \mathbf{GL}_1(C)$ is embedded as an open subvariety of the affine space $\mathbb{A}(C)$. By the classical homotopy invariance and localization, the pull-back homomorphism

$$\mathbb{Z} \cdot 1 = K_0\big(\mathbb{A}(C)\big) \to K_0(T)$$

is surjective. We have proved

29 **Proposition 29** For a quasi-trivial torus T, one has $K_0(T) = \mathbb{Z} \cdot 1$.

We generalize this statement in Theorem 30.

2.6.4 Coflasque Algebraic Tori

An algebraic torus T over F is called *coflasque* if for every field extension L/F the Galois cohomology group $H^1(L, T^*_{\text{sep}})$ is trivial, or equivalently, if $\operatorname{Pic}(T_L) = 0$. For example, quasi-trivial tori are coflasque.

30 **Theorem 30** Let T be a coflasque torus and let U be a principal homogeneous space of T. Then $K_0(U) = \mathbb{Z} \cdot 1$.

Proof Let X be a smooth projective toric model of T (for the existence of X see [1]). The variety $Y = T\backslash(X \times U)$ is then a toric variety of T that has an open orbit isomorphic to U.

By Theorem 21, there exist an étale F-algebra A, a separable F-algebra B of rank n^2 over its center A and morphisms $u : Y \to B$, $v : B \to Y$ in $\mathcal{C}$ such that $v \circ u = \text{id}$. The morphism u is represented by a locally free $\mathcal{O}_Y$-module in $\mathcal{P}(Y, B)$ of rank n. The class of the algebra B in $\operatorname{Br}(A)$ belongs to the image of $\lambda^{U_A} : \operatorname{Pic}(T_A) \to \operatorname{Br}(A)$. The torus T is coflasque, hence the group $\operatorname{Pic}(T_A)$ is trivial and therefore, the algebra B splits, $B \simeq M_n(A)$, so that $K_0(B^{\text{op}})$ is isomorphic canonically to $K_0(A)$.

Applying the realization functor to the morphism $u^{\text{op}} : B^{\text{op}} \to Y$ we get a (split) surjection

$$\mathcal{K}_0(u^{\text{op}}) : K_0(B^{\text{op}}) \to K_0(Y) .$$

Under the identification of $K_0(B^{\mathrm{op}})$ with $K_0(A)$ we get a (split) surjection

$$\mathcal{K}_0(w^{\mathrm{op}}) : K_0(A) \to K_0(Y) ,$$

where w is a certain element in $K_0(Y, A)$ represented by a locally free $\mathcal{O}_Y$-module of rank one, i.e., by an element of $\mathrm{Pic}(Y \otimes_F A)$.

It follows that $K_0(Y)$ is generated by the push-forwards of the classes of $\mathcal{O}_Y$-modules from $\mathrm{Pic}(Y_E)$ for all finite separable field extensions E/F. Since the pull-back homomorphism $K_0(Y) \to K_0(U)$ is surjective, the analogous statement holds for the open subset $U \subset Y$. But by [18, Prop. 6.10], there is an injection $\mathrm{Pic}(U_E) \hookrightarrow \mathrm{Pic}(T_E) = 0$, hence $\mathrm{Pic}(U_E) = 0$ and therefore $K_0(U) = \mathbb{Z} \cdot 1$.

Equivariant K-theory of some Reductive Groups 2.7

Spectral Sequence 2.7.1

Let G be a split reductive group over a field F. Choose a maximal split torus $T \subset G$.

Let X be a G-variety. The group $K'_n(G; X)$ (resp. $K'_n(T; X)$) is a module over the representation ring $R(G)$ (resp. $R(T)$). The restriction map $K'_n(G; X) \to K'_n(T; X)$ is an homomorphism of modules with respect to the restriction ring homomorphism $R(G) \to R(T)$ and hence it induces an $R(T)$-module homomorphism

$$\eta : R(T) \otimes_{R(G)} K'_n(G; X) \to K'_n(T; X) .$$

Proposition 31 Assume that the commutator subgroup of G is simply connected. Then the homomorphism η is an isomorphism. 31

Proof Let $B \subset G$ be a Borel subgroup containing T. Set $Y = G/B$. By Proposition 15, there is an isomorphism $u : F^m \xrightarrow{\sim} Y$ in the category $\mathcal{C}(G)$ defined by some elements $u_1, u_2, \ldots, u_m \in K_0(G; Y) = R(B)$ that form a basis of $R(B)$ over $R(G)$. Applying the realization functor (see Sect. 2.3)

$$\mathcal{K}_n^X : \mathcal{C}(G) \to \textbf{Abelian Groups} ,$$

to the isomorphism u, we obtain an isomorphism

$$K'_n(G; X)^m \xrightarrow{\sim} K'_n(G; X \times Y) .$$

Identifying $K'_n(G; X)^m$ with $R(B) \otimes_{R(G)} K'_n(G; X)$ using the same elements u_i we get a canonical isomorphism

$$R(B) \otimes_{R(G)} K'_n(G; X) \xrightarrow{\sim} K'_n(G; X \times Y) .$$

Composing this isomorphism with the canonical isomorphism (Corollary 5)

$$K'_n(G; X \times Y) \xrightarrow{\sim} K'_n(B; X) ,$$

and identifying $K'_n(B; X)$ with $K'_n(T; X)$ via the restriction homomorphism (Theorem 23) we get the isomorphism η.

Since $R(T)$ is free $R(G)$-module by [20, Th.1.3], in the assumptions of Proposition 31 one has

$$\operatorname{Tor}_p^{R(G)}(\mathbb{Z}, K'_n(G; X)) \simeq \operatorname{Tor}_p^{R(T)}(\mathbb{Z}, K'_n(T; X)) , \tag{2.2}$$

where we view $\mathbb{Z}$ as a $R(G)$-module via the *rank* homomorphism $R(G) \to \mathbb{Z}$.

Proposition 24 then yields:

32 **Theorem 32** [11, Th. 4.3] Let G be a split reductive group defined over F with the simply connected commutator subgroup and let X be a G-variety. Then there is a spectral sequence

$$E^2_{p,q} = \operatorname{Tor}_p^{R(G)}(\mathbb{Z}, K'_q(G; X)) \Rightarrow K'_{p+q}(X) .$$

33 **Corollary 33** The restriction homomorphism $K'_0(G; X) \to K'_0(X)$ induces an isomorphism $\mathbb{Z} \otimes_{R(G)} K'_0(G; X) \simeq K'_0(X)$.

In the smooth projective case, Proposition 28 and (2.2) give the following generalization of Corollary 33:

34 **Corollary 34** If X is a smooth projective G-variety, then the spectral sequence in Theorem 32 degenerates. i.e.,

$$\operatorname{Tor}_p^{R(G)}(\mathbb{Z}, K'_n(G; X)) = \begin{cases} K'_n(X) , & \text{if } p = 0 , \\ 0 , & \text{if } p > 0 . \end{cases}$$

2.7.2 K-theory of Simply Connected Group

The following technical statement is very useful.

35 **Proposition 35** Let G be an algebraic group over F and let $f : X \to Y$ be a G-torsor over F. For every point $y \in Y$ let X_y be the fiber $f^{-1}(y)$ of f over y (so that X_y is a principal homogeneous space of G over the residue field $F(y)$). Assume that $K_0(X_y) = \mathbb{Z} \cdot 1$ for every point $y \in Y$. Then the restriction homomorphism $K'_0(Y) \simeq K'_0(G; X) \to K'_0(X)$ is surjective.

Proof We prove that the restriction homomorphism $\mathrm{res}^X : K_0'(G;X) \to K_0'(X)$ is surjective by induction on the dimension of X. Assume that we have proved the statement for all varieties of dimension less than the dimension of X. We would like to prove that res^X is surjective.

We prove this statement by induction on the number of irreducible components of Y. Suppose first that Y is irreducible. By Corollary 9, we may assume that Y is reduced.

Let $y \in Y$ be the generic point and let $v \in K_0'(X)$. Since $K_0'(X_y) = K_0(X_y) = \mathbb{Z} \cdot 1$, the restriction homomorphism $K_0'(G;X_y) \to K_0'(X_y)$ is surjective. It follows that there exists a non-empty open subset $U' \subset Y$ such that the pull-back of v in $K_0'(U)$, where $U = f^{-1}(U')$, belongs to the image of the restriction homomorphism $K_0'(G;U) \to K_0'(U)$. Set $Z = X \setminus U$ (considered as a reduced closed subvariety of X). Since $\dim(Z) < \dim(X)$ and $Z \to Y \setminus U'$ is a G-torsor, by the induction hypothesis, the left vertical homomorphism in the commutative diagram with the exact rows

$$\begin{array}{ccccccc} K_0'(G;Z) & \xrightarrow{i_*} & K_0'(G;X) & \xrightarrow{j^*} & K_0'(G;U) & \to & 0 \\ \mathrm{res}^Z \downarrow & & \mathrm{res}^X \downarrow & & \mathrm{res}^U \downarrow & & \\ K_0'(Z) & \xrightarrow{i_*} & K_0'(X) & \xrightarrow{j^*} & K_0'(U) & \to & 0 \end{array}$$

is surjective. Hence, by diagram chase, $v \in \mathrm{im}(\mathrm{res}^X)$.

Now let Y be an arbitrary variety. Choose an irreducible component Z' of Y and set $Z = f^{-1}(Z')$, $U = X \setminus Z$. The number of irreducible components of U is less than one of X. By the first part of the proof and the induction hypothesis, the homomorphisms res_Z and res_U in the commutative diagram above are surjective. It follows that res^X is also surjective.

I. Panin has proved in [16] that for a principal homogeneous space X of a simply connected group of inner type, $K_0(X) = \mathbb{Z} \cdot 1$. In the next statement we extend this result to arbitrary simply connected groups (and later in Theorem 38 - to factorial groups).

Proposition 36 Let G be a simply connected group and let X be a principal homogeneous space of G. Then $K_0(X) = \mathbb{Z} \cdot 1$. 36

Proof Suppose first that G is a quasi-split group. Choose a maximal torus T of a Borel subgroup B of G. A fiber of the projection $f : T\backslash X \to B\backslash X$ is isomorphic to the unipotent radical of B and hence is isomorphic to an affine space. By [17, §7, Prop. 4.1], the pull-back homomorphism

$$f^* : K_0(B\backslash X) \to K_0(T\backslash X)$$

is an isomorphism.

The character group T^* is generated by the fundamental characters and therefore, T^* is a permutation Galois module, so that T is a quasi-trivial torus. Every principal homogeneous space of T is trivial, hence by Propositions 29 and 35, the restriction homomorphism

$$K_0(T\backslash X) = K_0(T; X) \to K_0(X)$$

is surjective. Thus, the pull-back homomorphism $g^* : K_0(B\backslash X) \to K_0(X)$ with respect to the projection $g : X \to B\backslash X$ is surjective.

Let G_1 be the algebraic group of all G-automorphisms of X. Over F_{sep}, the groups G and G_1 are isomorphic, so that G_1 is a simply connected group. The variety X can be viewed as a G_1-torsor [10, Prop. 1.2]. In particular, $B\backslash X$ is a projective homogeneous variety of G_1.

In the commutative diagram

$$\begin{array}{ccc} K_0(G_1; B\backslash X) & \rightarrow & K_0(G_1; X) \\ \text{res}\downarrow & & \text{res}\downarrow \\ K_0(B\backslash X) & \overset{g^*}{\rightarrow} & K_0(X) \end{array}$$

the left vertical homomorphism is surjective by Corollary 17. Since g^* is also surjective, so is the right vertical restriction. It follows from Proposition 3 that

$$K_0(G_1; X) = K_0(\operatorname{Spec} F) = \mathbb{Z} \cdot 1 ,$$

hence, $K_0(X) = \mathbb{Z} \cdot 1$.

Now let G be an arbitrary simply connected group. Consider the projective homogeneous variety Y of all Borel subgroups of G. For every point $y \in Y$, the group $G_{F(y)}$ is quasi-split. The fiber of the projection $X \times Y \to Y$ over y is the principal homogeneous space $X_{F(y)}$ of $G_{F(y)}$. By the first part of the proof, $K_0(X_{F(y)}) = \mathbb{Z} \cdot 1$. Hence by Proposition 35, the pull-back homomorphism

$$K_0(Y) \to K_0(X \times Y)$$

is surjective. It follows from Example 14 that the natural homomorphism $\mathbb{Z} \cdot 1 = K_0(F) \to K_0(X)$ is a direct summand of this surjection and therefore, is surjective. Therefore, $K_0(X) = \mathbb{Z} \cdot 1$.

2.8 Equivariant K-theory of Factorial Groups

An algebraic group G over a field F is called *factorial* if for any finite field extension E/F the Picard group $\operatorname{Pic}(G_E)$ is trivial.

37 **Proposition 37** [11, Prop. 1.10] A reductive group G is factorial if and only if the commutator subgroup G' of G is simply connected and the torus G/G' is coflasque.

In particular, simply connected groups and coflasque tori are factorial.

Theorem 38 Let G be a factorial group and let X be a principal homogeneous space of G. Then $K_0(X) = \mathbb{Z} \cdot 1$. 38

Proof Let G' be the commutator subgroup of G and let $T = G/G'$. The group G' is simply connected and the torus T is coflasque. The variety X is a G'-torsor over $Y = G' \backslash X$. By Propositions 35 and 36, the restriction homomorphism

$$K_0(Y) = K_0(G'; X) \to K_0(X)$$

is surjective. The variety Y is a principal homogeneous space of T and by Theorem 30, $K_0(Y) = \mathbb{Z} \cdot 1$, whence the result.

Theorem 39 [11, Th. 6.4] Let G be a reductive group defined over a field F. Then the following condition are equivalent: 39

1. G is factorial.
2. For every G-variety X, the restriction homomorphism

$$K'_0(G; X) \to K'_0(X)$$

is surjective.

Proof (1) $\Rightarrow$ (2). Consider first the case when there is a G-torsor $X \to Y$. Then the restriction homomorphism $K_0(G; X) \to K_0(X)$ is surjective by Proposition 35 and Theorem 38.

In the general case, choose a faithful representation $G \hookrightarrow S = \mathbf{GL}(V)$. Let $\mathbb{A}$ be the affine space of the vector space $\mathrm{End}(V)$ so that S is an open subvariety in $\mathbb{A}$. Consider the commutative diagram

$$\begin{array}{ccccc} K'_0(G; X) & \xrightarrow{\sim} & K'_0(G; \mathbb{A} \times X) & \to & K'_0(G; S \times X) \\ \mathrm{res} \downarrow & & \mathrm{res} \downarrow & & \mathrm{res} \downarrow \\ K'_0(X) & \xrightarrow{\sim} & K'_0(\mathbb{A} \times X) & \to & K'_0(S \times X) \end{array} .$$

The group G acts freely on $S \times X$ so that we have a G-torsor $S \times X \to Y$. In fact, Y exists in the category of algebraic spaces and may not be a variety. One should use the equivariant K'-groups of algebraic spaces as defined in [21]. By the first part of the proof, the right vertical map is surjective. By localization, the right horizontal arrows are the surjections. Finally, the composition in the bottom row is an isomorphism since it has splitting $K'_0(S \times X) \to K'_0(X)$ by the pull-back with respect to the closed embedding $X = \{1\} \times X \hookrightarrow S \times X$ of finite Tor-dimension (see [17, §7, 2.5]). Thus, the left vertical restriction homomorphism is surjective.

(2) $\Rightarrow$ (1). Taking $X = G_E$ for a finite field extension E/F, we have a surjective homomorphism

$$\mathbb{Z} \cdot 1 = K_0(E) = K_0(G; G_E) \to K_0(G_E) ,$$

i.e. $K_0(G_E) = \mathbb{Z} \cdot 1$. Hence, the first term of the topological filtration $K_0(G_E)^{(1)}$ of $K_0(G_E)$ (see [17, §7.5]), that is the kernel of the rank homomorphism $K_0(G_E) \to \mathbb{Z}$, is trivial. The Picard group $\mathrm{Pic}(G_E)$ is a factor group of $K_0(G_E)^{(1)}$ and hence is also trivial, i.e., G is a factorial group.

In the end of the section we consider the smooth projective case.

40 **Theorem 40** [11, Th. 6.7] Let G be a factorial reductive group and let X be a smooth projective G-variety over F. Then the restriction homomorphism

$$K'_n(G; X) \to K'_n(X)$$

is split surjective.

Proof Consider the smooth variety $X \times X$ with the action of G given by $g(x, x') = (x, gx')$. By Theorem 39, the restriction homomorphism $K'_0(G; X \times X) \to K'_0(X \times X)$ is surjective. Hence by Lemma 25, applied to the trivial subgroup of G, the restriction homomorphism $K'_n(G; X) \to K'_n(X)$ is a split surjection.

2.9 Applications

2.9.1 K-theory of Classifying Varieties

Let G be an algebraic group over a field F. Choose a faithful representation $\mu : G \hookrightarrow \mathbf{GL}_n$ and consider the factor variety $X = \mathbf{GL}_n / \mu(G)$. For every field extension E/F, the set $H^1(E, G)$ of isomorphism classes of principal homogeneous spaces of G over E can be identified with the orbit space of the action of $\mathrm{GL}_n(E)$ on $X(E)$ [7, Cor. 28.4]:

$$H^1(E, G) = \mathrm{GL}_n(E) \backslash X(E) .$$

The variety X is called a *classifying variety of* G. The $\mathrm{GL}_n(E)$-orbits in the set $X(E)$ classify principal homogeneous spaces of G over E.

We can compute the Grothendieck ring of a classifying variety X of G. M. Rost used this result for the computation of orders of the Rost's invariants (see [5]). As shown in Example 4, the G-torsor $\mathbf{GL}_n \to X$ induces the homomorphism $r : R(G) \to K_0(X)$ taking the class of a finite dimensional representation $\varrho : G \to \mathbf{GL}(V)$ to the class the vector bundle E_ϱ.

41 **Theorem 41** Let X be a classifying variety of an algebraic group G. The homomorphism r gives rise to an isomorphism

$$\mathbb{Z} \otimes_{R(\mathbf{GL}_n)} R(G) \simeq K_0(X) .$$

In particular, the group $K_0(X)$ is generated by the classes of the vector bundles E_ϱ for all finite dimensional representations ϱ of G over F.

Proof The Corollary 33 applied to the smooth $\mathbf{GL}_n$-variety X yields an isomorphism

$$\mathbb{Z} \otimes_{R(\mathbf{GL}_n)} K_0(\mathbf{GL}_n; X) \simeq K_0(X) .$$

On the other hand,

$$K_0(\mathbf{GL}_n; X) \simeq R(G)$$

by Corollary 6.

Note that the structure of the representation ring of an algebraic group is fairly well understood in terms of the associated root system and indices of the Tits algebras of G (see [22], [5, Part 2, Th. 10.11]).

Equivariant Chow Groups 2.9.2

For a variety X over a field F we write $\mathrm{CH}_i(X)$ for the *Chow group* of equivalence classes of dimension i cycles on X [4, I.1.3]. Let G be an algebraic group G over F. For X a G-variety, D. Edidin and W. Graham have defined in [3] the *equivariant Chow groups* $\mathrm{CH}_i^G(G)$. There is an obvious *restriction* homomorphism

$$\mathrm{res} : \mathrm{CH}_i^G(X) \to \mathrm{CH}_i(X) .$$

Theorem 42 Let X be a G-variety of dimension d, where G is a factorial group. Then the restriction homomorphism 42

$$\mathrm{res} : \mathrm{CH}_{d-1}^G(X) \to \mathrm{CH}_{d-1}(X)$$

is surjective.

Proof The proof is essentially the same as the one of Theorem 39. We use the homotopy invariance property and localization for the equivariant Chow groups. In the case of a torsor the proof goes the same lines as in Proposition 35. The only statement to check is the triviality of $\mathrm{CH}^1(Y) = \mathrm{Pic}(Y)$ for a principal homogeneous space Y of G. By [18, Prop. 6.10], the group $\mathrm{Pic}(Y)$ is isomorphic to a subgroup of $\mathrm{Pic}(G)$, which is trivial since G is a factorial group.

Let $\mathrm{Pic}^G(X)$ denote the group of line G-bundles on X. If X is smooth irreducible, the natural homomorphism $\mathrm{Pic}^G(X) \to \mathrm{CH}_{d-1}^G(X)$ is an isomorphism [3, Th. 1].

43 **Corollary 43** (Cf. [14, Cor. 1.6]) Let X be a smooth G-variety, where G is a factorial group. Then the restriction homomorphism

$$\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$$

is surjective. In other words, every line bundle on X has a structure of a G-vector bundle.

2.9.3 Group Actions on the K'-groups

Let G be an algebraic group and let X be a G-variety over F. For every element $g \in G(F)$ write λ_g for the automorphism $x \mapsto gx$ of X. The group $G(F)$ acts naturally on $K'_n(X)$ by the pull-back homomorphisms λ_g^*.

44 **Theorem 44** [11, Prop.7.20] Let G be a reductive group and let X be a G-variety. Then

1. The group $G(F)$ acts trivially on $K'_0(X)$.
2. If X is smooth and projective, the group $G(F)$ acts trivially on $K'_n(X)$ for every $n \geq 0$.

Proof By [11, Lemma 7.6], there exists an exact sequence

$$1 \to P \to \widetilde{G} \xrightarrow{\pi} G \to 1$$

with a factorial reductive group $\widetilde{G}$ and a quasi-trivial torus P. It follows from the exactness of the sequence

$$\widetilde{G}(F) \xrightarrow{\pi(F)} G(F) \to H^1(F, P(F_{\mathrm{sep}}))$$

and triviality of $H^1(F, P(F_{\mathrm{sep}}))$ (Hilbert Theorem 90) that the homomorphism $\pi(F) : \widetilde{G}(F) \to G(F)$ is surjective. Hence, we can replace G by $\widetilde{G}$ and assume that G is factorial.

By definition of a G-module M, the isomorphism

$$\varrho : \theta^*(M) \xrightarrow{\sim} p_2^*(M) ,$$

where $\theta : G \times X \to X$ is the action morphism, induces an isomorphism of two compositions $\theta^* \circ \mathrm{res}$ and $p_2^* \circ \mathrm{res}$ in the diagram

$$\mathcal{M}(G;X) \xrightarrow{\mathrm{res}} \mathcal{M}(X) \underset{p_2^*}{\overset{\theta^*}{\rightrightarrows}} \mathcal{M}(G \times X) .$$

Hence the compositions

$$K'_n(G;X) \xrightarrow{\mathrm{res}} K'_n(X) \underset{p_2^*}{\overset{\theta^*}{\rightrightarrows}} K'_n(G \times X)$$

are equal.

For any $g \in G(F)$ write ε_g for the morphism $X \to G \times X$, $x \mapsto (g, x)$. Then clearly $p_2 \circ \varepsilon_g = \mathrm{id}_X$ and $\theta \circ \varepsilon_g = \lambda_g$. The pull-back homomorphism ε_g^* is defined since ε_g is of finite Tor-dimension [17, §7, 2.5]. Thus, we have $\varepsilon_g^* \circ p_2^* = \mathrm{id}$ and $\varepsilon_g^* \circ \theta^* = \lambda_g^*$ on $K'_n(X)$, hence

$$\mathrm{res} = \varepsilon_g^* \circ p_2^* \circ \mathrm{res} = \varepsilon_g^* \circ \theta^* \circ \mathrm{res} = \lambda_g^* \circ \mathrm{res} : K'_n(G; X) \to K'_n(X) .$$

By Theorem 39, the restriction homomorphism res is surjective for $n = 0$, hence $\lambda_g^* = \mathrm{id}$. In the case of smooth projective X the restriction is surjective for every $n \geq 0$ (Theorem 40), hence again $\lambda_g^* = \mathrm{id}$.

Corollary 45 Let G be a reductive group and let X be a smooth G-variety. Then the group $G(F)$ acts trivially on $\mathrm{Pic}(X)$. 45

Proof The Picard group $\mathrm{Pic}(X)$ is isomorphic to a subfactor of $K_0(X)$ and $G(F)$ acts trivially on $K_0(X)$ by Theorem 44.

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