Solutions for the Midterm

1) We consider the augmented coefficient matrix of the linear system and compute its reduced row-echelon form:

$$A = \left(\begin{array}{cccc|c} 0 & 0 & 1 & 2 & 1 \\ 3 & 3 & 5 & 4 & 2 \\ 1 & 1 & 2 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 1 \\ 3 & 3 & 5 & 4 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

rref(A)

pivot variables : x_1, x_3

free variables : x_2, x_4

$x_2 = s$, $\quad x_1 = -x_2 + 2x_4 - 1 = -s + 2t - 1$

$x_4 = t$ $\quad x_3 = -2x_4 + 1 = -2t + 1$

So the solutions of the linear system are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s+2t-1 \\ s \\ -2t+1 \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix},$$

where $s, t \in \mathbb{R}$ are arbitrary.

(2)

2) $\left(\begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array}\right)$

$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 0 & 1 \\ 0 & 7 & 2 & 0 & 1 & 2 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array}\right)$

$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 0 & \frac{1}{7} & 1 & -\frac{3}{7} & -\frac{6}{7} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 0 & 1 & 7 & -3 & -6 \end{array}\right)$

$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & 7 & -3 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -7 & 3 & 7 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & 7 & -3 & -6 \end{array}\right)$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & 7 & -3 & -6 \end{array}\right)$

$\underbrace{\qquad\qquad}_{\text{rref}(A).}$

Since rref $(A) = I_3$, the matrix A is invertible. Moreover, A^{-1} is given by

$$A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ -2 & 1 & 2 \\ 7 & -3 & -6 \end{pmatrix}.$$

③

3) a) The vectors $v_1, \dots, v_k$ in a subspace V of $\mathbb{R}^n$ span this subspace if every vector in V can be expressed as a linear combination of the vectors $v_1, \dots, v_k$.

b) The vectors $v_1, \dots, v_k$ are linearly independent if the zero vector can be expressed as a linear combination of the vectors $v_1, \dots, v_k$ only in a trivial way, that is, the only solution of the equation

$$c_1 v_1 + \dots + c_k v_k = 0$$

for the scalars $c_1, \dots, c_k$ in $\mathbb{R}$ is given by $c_1 = c_2 = \dots = c_k = 0$.

c) The vectors $v_1, \dots, v_k$ in $\mathbb{R}^n$ form a basis of a subspace V of $\mathbb{R}^n$, if they are linearly independent and span V.

d) The vectors span $\mathbb{R}^2$, because every vector in $\mathbb{R}^2$ can be expressed as a linear combination of v_1, v_2, v_3; indeed, if $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, then $\begin{pmatrix} x \\ y \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x v_1 + y v_3$.

The vectors are not linearly independent, because we have

(4) $-2 \cdot v_1 + v_2 - v_3 = -2 \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}$

$$= \begin{pmatrix}0\\0\end{pmatrix} = 0,$$

and so the zero vector $0 \in \mathbb{R}^2$ can be represented as a linear combination of the vectors v_1, v_2, v_3 in a non-trivial way.

The vectors v_1, v_2, v_3 do not form a basis of $\mathbb{R}^2$, because they are not linearly independent.

4) a) We check the three criteria for V to be a subspace of $\mathbb{R}^n$:

i) $0 \in V$, because $A0 = 0$.

ii) if $x, y \in V$, then $Ax = x$ and $Ay = y$.

Then $A(x+y) = Ax + Ay = x+y$.

So $x+y \in V$, and V is closed under vector addition.

iii) if $x \in V$ and $c \in \mathbb{R}$, then

$A(cx) = c(Ax) = cx$.

So $cx \in V$, and V is closed under scalar multiplication.

We conclude that V is a subspace of $\mathbb{R}^n$.

b) Note that in (a) we have $Ax = x$ precisely if $(A - I_n)x = Ax - x = 0$.

(5) So V is the null-space $\mathcal{N}(B)$ of the matrix $B = A - I_n$.

For the specific matrix A given, we have $n=3$ and

$$B = A - I_3 = \begin{pmatrix} 1 & 3 & 2 \\ -2 & -6 & -4 \\ 3 & 9 & 6 \end{pmatrix}.$$

To find a basis of $\mathcal{N}(B)$, we compute rref(B):

$$\begin{pmatrix} 1 & 3 & 2 \\ -2 & -6 & -4 \\ 3 & 9 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ rref}(B).$$

For the corresponding homogeneous system we have the pivot variables: x_1

free variables: x_2, x_3.

$x_2 = s$, $x_3 = t$

$$x_1 = -3x_2 - 2x_3 = -3s - 2t.$$

So $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3s-2t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

is the general solution of the homogeneous system with cofficient matrix B.

We conclude that

$$v_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$ form a basis of $V = \mathcal{N}(B)$.

(6) 5) a) Suppose A is an $(n \times n)$-matrix such that $Ax = x$ for all $x \in \mathbb{R}^n$.
If (as usual)

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

denote the standard basis vectors in $\mathbb{R}^n$, then $Ae_k = e_k$ for $k = 1, \ldots, n$.
On the other hand, let $c_1, \ldots, c_n$ be the column vectors of A.
Then

$$e_k = A e_k = (c_1 c_2 \cdots c_n) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k\text{-th position}$$

$$= c_k \quad \text{for each } k \in \{1, \ldots, n\}.$$

So the columns of A are given by the standard basis vectors $e_1, \ldots, e_n$, and we conclude that

$$A = (c_1 \cdots c_n) = (e_1 \cdots e_n) = I_n$$

as desired. □

b) Let $A = B^3$. We claim that A satisfies the condition in (a) (for $n = 3$).
To see this, first note that

$$Av_1 = B^3 v_1 = B(B(Bv_1)) = B(Bv_2) = Bv_3 = v_1$$

Similarly, $Av_2 = v_2$ and $Av_3 = v_3$. Now let $x \in \mathbb{R}^3$ be arbitrary. Since v_1, v_2, v_3 form a basis of $\mathbb{R}^3$, the vector

(7) x can be expressed as a linear combination of the vectors v_1, v_2, v_3. So there exist scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3.$$

It follows that

$$\begin{aligned} Ax &= A(c_1 v_1 + c_2 v_3 + c_3 v_3) \\ &= c_1 A v_1 + c_2 A v_2 + c_3 A v_3 \\ &= c_1 v_1 + c_2 v_2 + c_3 v_3 = x. \end{aligned}$$

So indeed $Ax = x$ for all $x \in \mathbb{R}^3$, and we conclude by (a) that $B^3 = A = I_3$ as desired. $\square$