Part of Homework 6 (Due: Fr, 2/22)

In each of the following problems, give a clear explanation of your reasoning for the solution. Remember: Mere computations without running commentary=0 points.

Problem 1: Let $v_1, \dots, v_k$ be orthonormal vectors in $\mathbb{R}^n$; so they satisfy $(v_i \cdot v_j) = \delta_{ij}$ for $i, j = 1, \dots k$, where δ_{ij} is the Kronecker delta. Show that then the vectors $v_1, \dots, v_k$ are linearly independent. (5 pts)

Problem 2: Let V be a subspace of $\mathbb{R}^n$ and $V^\perp$ be its orthogonal complement. We choose a basis $v_1, \dots, v_k$ of V, and a basis $u_1, \dots, u_l$ of $V^\perp$.

a) Show that the vectors $v_1, \dots, v_k, u_1, \dots, u_l$ are linearly independent. (4 pts)

b) Show that the vectors $v_1, \dots, v_k, u_1, \dots, u_l$ form a basis of $\mathbb{R}^n$. (2 pts)

c) Show that every vector $x \in \mathbb{R}^n$ can be uniquely represented as $x = p + p^\perp$, where $p \in V$ and $p^\perp \in V^\perp$. (4 pts)

Problem 3: Let V be a subspace of $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and $p = \text{proj}_V(x)$ be the orthogonal projection of x into V.

a) Show that if $v \in V$, then $\|x - v\|^2 = \|x - p\|^2 + \|p - v\|^2$. (5 pts)

b) Show that p is the unique vector in p that minimizes the distance to x, that is, $\|x - p\| \leq \|x - v\|$ for all $v \in V$, with equality iff $v = p$. (5 pts)

Problem 4: Let V be a subspace of $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and $p = \text{proj}_V(x)$. Show that p is the unique vector in V such that $x - p \perp V$. Hint: Suppose v is a vector in V with $x - v \perp V$. Show that then $(p - v) \cdot (p - v) = 0$. (5 pts)