1

# Solutions for the Final Exam

1) Since $\ker(A) = \{x \in \mathbb{R}^4 : Ax = 0\}$, we first solve the homogeneous system $Ax = 0$.

$$A = \begin{pmatrix} 1 & 2 & -2 & -3 \\ -1 & -2 & 1 & 4 \\ 1 & 2 & -3 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -2 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -2 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \underbrace{\begin{pmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{rref}(A)}$$

free variables $x_2 = s$, $x_4 = t$.

$x_1 = -2x_2 + 5x_4 = -2s + 5t$,

$x_3 = x_4 = t$.

General solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2s+5t \\ s \\ t \\ t \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad s,t \in \mathbb{R} \text{ arb.}$$

It follows that the vectors

$$v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 5 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$ form a basis of $\ker(A)$.

(2) To find an orthonormal basis of $\ker(A)$, we apply the Gram-Schmidt orthonormalization process.

$$u_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$u_2' = v_2 - (v_2 \cdot u_1) u_1 = \begin{pmatrix} 5 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{5} \cdot (-10) \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{\|u_2'\|} u_2' = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}.$$

Conclusion:

$u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ form an ONB of $\ker(A)$.

2) a)

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 8 \\ 1 & 1 & 2 & 3 \\ 1 & 3 & 5 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 2 & 4 \\ -1 & -1 & -1 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ -1 & -1 & -1 \\ 1 & 2 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = \underline{\underline{-2}}.$$

b) $A$ is invertible, because $\det(A) \neq 0$.

c) $\det(A^{-1} A^T A A^T) = -\frac{1}{2} \cdot (-2) \cdot (-2) \cdot (-2) = \underline{\underline{4}}.$

③

d) If $y \in \mathbb{R}^4$ is arb., then $y = A(A^{-1}y)$
$= Ax$, where $x = A^{-1}y$.
So every vector $y \in \mathbb{R}^4$ can be written in the form $y = Ax$ with $x \in \mathbb{R}^4$.

3) We first compute the eigenvalues of $A$ by finding the roots of the characteristic polynomial $p_A(\lambda)$ of $A$.

$$p_A(\lambda) = \begin{vmatrix} -\lambda & 2 & 0 \\ 0 & 2-\lambda & 0 \\ 2 & -4 & 1-\lambda \end{vmatrix} = (2-\lambda)\cdot \begin{vmatrix} -\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-\lambda)(1-\lambda).$$

So the eigenvalues of $A$ are
$\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$.
Since the $3\times 3$-matrix $A$ has three distinct real eigenvalues, it is diagonalizable.
To find the matrix $S$, we compute an eigenbasis of $\mathbb{R}^3$ for $A$.

$$E_2: \begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & -4 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 2 & -4 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \text{ rref.}$$

free variable $x_3 = s$, $x_1 = -\frac{1}{2}x_3 = -\frac{1}{2}s$,
$x_2 = -\frac{1}{2}x_3 = -\frac{1}{2}s$.

(4)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} = s' \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \quad s, s' \in \mathbb{R} \text{ arb.}$$

$v_1 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ is a basis of $E_2$.

$$E_1: \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ rref.}$$

free variable: $x_3 = s$

$x_1 = x_2 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R} \text{ arb.}$$

$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a basis of $E_1$.

$$E_0: \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 2 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ rref.}$$

free variable $x_3 = s$.

$x_1 = -\frac{1}{2} x_3 = -\frac{1}{2} s$, $x_2 = 0$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} = s' \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad s, s' \in \mathbb{R} \text{ arb.}$$

(5) $V_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ is a basis of $E_0$.

Let $S = (V_1\ V_2\ V_3) = \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix}$ and

$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A = SDS^{-1}$.

4)

$$\begin{array}{cccc} 1 & x & x^2 & \\ \end{array}\qquad y$$

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} 7 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

To find $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ we solve the normal equation $A^T A u = A^T b$.

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{pmatrix}.$$

$A^T b = \begin{pmatrix} 10 \\ -10 \\ 34 \end{pmatrix}$. The augmented coefficient matrix is

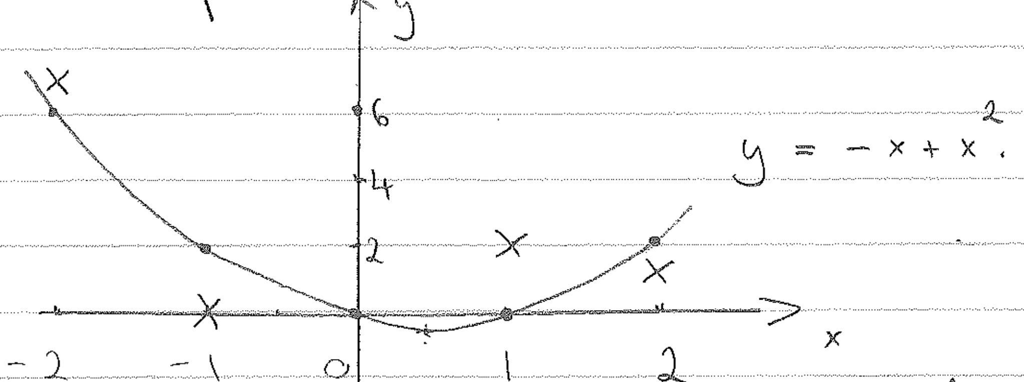
$$\left(\begin{array}{ccc|c} 4 & 0 & 10 & 10 \\ 0 & 10 & 0 & -10 \\ 10 & 0 & 34 & 34 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 9 & 9 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \text{So } u = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

(6)

So $f(x) = -x + x^2$ gives the best least square fit to the data.

b)




data points and graph of function giving best fit.

5) We first find the singular values of $A$ by computing the eigenvalues of $A^TA$.

$$A^TA = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}.$$

$$P_A(\lambda) = \begin{vmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix} \quad \text{(Sarrus!)}$$

$$= (10-\lambda)^2(2-\lambda) - \left(4\cdot(10-\lambda) + 16(10-\lambda)\right)$$

$$= (10-\lambda)\left[20 - 12\lambda + \lambda^2 - 20\right]$$

$$= (10-\lambda)\,\lambda\,(\lambda - 12).$$

So the eigenvalues of $A^TA$ are

$\lambda_1 = 12$, $\lambda_2 = 10$, $\lambda_3 = 0$.

7

Thus, A has the singular values
$\sigma_1 = \sqrt{\lambda_1} = \sqrt{12}$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{10}$, $\sigma_3 = \sqrt{\lambda_3} = 0$.
We compute an orthonormal eigenbasis for $A^T A$.

$$E_{12}: \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 4 & -8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$ rref $x_3 = s$ free variable
$x_1 = x_3 = s$, $x_2 = 2x_3 = 2s$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \; s \in \mathbb{R}.$$

$V_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is an ONB of $E_{12}$.

$$E_{10}: \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ rref $x_2 = s$ free variable
$x_1 = -2x_2 = -2s$
$x_3 = 0$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \; s \in \mathbb{R}.$$ $V_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ is an ONB of $E_{10}$.

(8) $E_0$ : $\begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 4 & \frac{8}{5} \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix}$ rref $x_3 = s$ free variable.

$x_1 = -\frac{1}{5}x_3 = -\frac{1}{5}s$.

$x_2 = -\frac{2}{5}x_3 = -\frac{2}{5}s$.

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} -1/5 \\ -2/5 \\ 1 \end{pmatrix} = s'\begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$, $s, s' \in \mathbb{R}$ arb.

$v_3 = \frac{1}{\sqrt{30}}\begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$ is an ONB of $E_0$.

$v_1, v_2, v_3$ form an orthonormal eigenbasis of $\mathbb{R}^3$ for $A^TA$.

The matrix $V$ has the columns $v_1, v_2, v_3$.

$u_1 = \frac{1}{\sigma_1}Av_1 = \frac{1}{\sqrt{12}}\frac{1}{\sqrt{6}}\begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

$u_2 = \frac{1}{\sigma_2}Av_2 = \frac{1}{\sqrt{10}}\frac{1}{\sqrt{5}}\begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$u_1, u_2$ form an ONB of $\mathbb{R}^2$ and so $U$ is the matrix with columns $u_1, u_2$.

Conclusion: The singular value decomposition of $A$ is given by

(9) $A = U \Sigma V^T$, where

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}.$$

6) a) A number $\lambda \in \mathbb{R}$ is called an eigenvalue of $A$ if there exists a vector $v \in \mathbb{R}^n$, $v \neq 0$, such that
$$Av = \lambda v.$$

b) The eigenspace $E_\lambda$ consists of all vectors $v \in \mathbb{R}^n$ such that
$$Av = \lambda v.$$

c) The vectors $v_1, \ldots, v_n$ form an eigenbasis of $\mathbb{R}^n$ for $A$ if each of the vectors is an eigenvector of $A$ and if they form a basis of $\mathbb{R}^n$.

d) Since $\lambda = 2$ is an eigenvalue of $A$, there ex. $v \in \mathbb{R}^n$, $v \neq 0$, such that $Av = 2v$.

Then
$$Bv = (A^2 + A)v = (2^2 + 2)v = 6v,$$
and so 6 is an eigenvalue for $B$.

(10)

7) a) An $n \times n$-matrix $A$ is orthogonal if and only if
$$\|Ax\| = \|x\| \quad \text{for all } x \in \mathbb{R}^n.$$

b) A square matrix $A$ is orthogonal if and only if $A$ is invertible and $A^{-1} = A^T$.

c) We use characterization (b): If $A$ and $B$ are orthogonal, then $A$ and $B$ are invertible and
$$A^{-1} = A^T, \quad B^{-1} = B^T.$$

Hence $AB$ is invertible and
$$(AB)^{-1} = B^{-1}A^{-1} = B^T A^T = (AB)^T.$$

It follows that $AB$ is orthogonal.

d) Suppose $\lambda$ is a (real) eigenvalue of an orthogonal matrix $A$. Then there exists a vector $v \in \mathbb{R}^n$, $v \neq 0$, such that $Av = \lambda v$.
It follows that
$$0 \neq \|v\| = \|Av\| = \|\lambda v\| = |\lambda| \, \|v\|,$$
and so $|\lambda| = 1$. We conclude that
$\lambda = 1$ or $\lambda = -1$.

(11)

8) To justify the "if and only if"-statement, we have to prove two implications.

"$\Longrightarrow$" Suppose the vectors $u_1, \dots, u_k, v_1, \dots, v_l$ are linearly independent.

WTS $U \cap V = \{0\}$.

To see this, let $x \in U \cap V$ be arbitrary. Since $u_1, \dots, u_k$ form a basis of $U$ and $x \in U$, there ex. scalars $a_1, \dots, a_k$ s.t.

$x = a_1 u_1 + \dots + a_k u_k$.

Similarly, there ex. scalars $b_1, \dots, b_l \in \mathbb{R}$ s.t.

$x = b_1 v_1 + \dots + b_l v_l$.

Then

$$0 = x - x = a_1 u_1 + \dots + a_k u_k + (-b_1) v_1 + \dots + (-b_l) v_l.$$

Since $u_1, \dots, u_k, v_1, \dots, v_l$ are linearly independent, it follows that

$a_1 = \dots = a_k = -b_1 = \dots = -b_l = 0$.

In particular,

$x = a_1 u_1 + \dots + a_k u_k = 0$.

So $x = 0$ is the only vector that belongs to both $U$ and $V$.

(Note that $0$ indeed belongs to both $U$ and $V$, because $U$ and $V$ are subspaces of $\mathbb{R}^n$).

(12) "$\Leftarrow$" Suppose $U \cap V = \{0\}$.

<u>WTS</u> $u_1, \ldots, u_k, v_1, \ldots, v_l$ are linearly independent.

Assume that

(*) $a_1 u_1 + \cdots + a_k u_k + b_1 v_1 + \cdots + b_l v_l = 0$

for some scalars $a_1, \ldots, a_k, b_1, \ldots, b_l \in \mathbb{R}$.

Define

$x := a_1 u_1 + \cdots + a_k u_k$.

Then

$x \in \operatorname{span}(u_1, \ldots, u_k) = U$.

Moreover,

$$x = a_1 u_1 + \cdots + a_k u_k = (-b_1) v_1 + \cdots + (-b_l) v_l \in \operatorname{span}(v_1, \ldots, v_l) = V.$$

Hence $x \in U \cap V$, and so $x = 0$.

Since $u_1, \ldots, u_k$ form a basis of $U$, these vectors are linearly independent. Since

$x = 0 = a_1 u_1 + \cdots + a_k u_k$,

it follows that $a_1 = \cdots = a_k = 0$.

Similarly, since $v_1, \ldots, v_l$ form a basis of $V$ and

$0 = -x = b_1 v_1 + \cdots + b_l v_l$, it follows that $b_1 = \cdots = b_l = 0$.

So we see that the equation (*) implies that $a_1 = \cdots = a_k = b_1 = \cdots = b_l = 0$.

Therefore, the vectors $u_1, \ldots, u_k, v_1, \ldots, v_l$ are linearly independent.

(13)

9) Fix $c \in \mathbb{R}$, $c \neq 1$. Define

$$A = I_n - cP \text{ and } B = I_n - \frac{c}{c-1}P.$$

Since $P^2 = P$, we have

$$AB = (I_n - cP)\left(I_n - \frac{c}{c-1}P\right)$$
$$= I_n - cP - \frac{c}{c-1}P + \frac{c^2}{c-1}P^2$$
$$= I_n + \underbrace{\left(-c - \frac{c}{c-1} + \frac{c^2}{c-1}\right)}_{0}P = I_n.$$

(Note that $-c - \frac{c}{c-1} + \frac{c^2}{c-1} = -c + \frac{c(c-1)}{c-1} = 0$).
It follows that $A = I_n - cP$ is invertible and

$$A^{-1} = B = I_n - \frac{c}{c-1}P.$$

10) Suppose $A$ is a symmetric $n\times n$-matrix with $A^3 = I_n$. Then $A$ is diagonalizable and so there ex. a diagonal matrix $D$ and an invertible matrix $S$ s.t. $A = SDS^{-1}$.

Then $I_n = A^3 = (SDS^{-1})(SDS^{-1})(SDS^{-1})$
$$= SD^3S^{-1}.$$

Hence $D^3 = S^{-1}I_nS = I_n$.

It follows that for each of the diagonal entries $\lambda$ of $D$ we have $\lambda^3 = 1$, and so $\lambda = 1$. It follows that $D = I_n$, which implies $A = SDS^{-1} = SS^{-1} = I_n$. $\square$