Homework 4 (due: Mo, May 14)

Problem 1: a) Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Show that then there exist functions $f_k \in C_c^\infty(\mathbb{R}^n)$ for $k \in \mathbb{N}$ such that

$$\|f_k - f\|_1 \to 0 \text{ and } \|f_k - f\|_2 \to 0$$

as $k \to \infty$.

b) Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Show that then

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g} \text{ and } \widehat{f \cdot g} = \widehat{f} * \widehat{g}.$$

Problem 2: Let μ be a complex Borel measure on $\mathbb{R}^n$. Then its Fourier transform $\widehat{\mu}\colon \mathbb{R}^n \to \mathbb{C}$ is defined as

$$\widehat{\mu}(\xi) = \int e^{-i\xi\cdot x}\, d\mu(x), \quad \xi \in \mathbb{R}^n.$$

a) Show that $\widehat{\mu}$ is a bounded continuous function, but in general we do not have $\widehat{\mu} \in C_0(\mathbb{R}^n)$.

Let ν be another complex Borel measure on $\mathbb{R}^n$.

b) Show that $\displaystyle\int \widehat{\nu}\, d\mu = \int \widehat{\mu}\, d\nu$.

c) Show that if $\widehat{\mu} = \widehat{\nu}$, then $\mu = \nu$. Hint: Consider the special case of (b), where one of the measures is absolutely continuous with respect to Lebesgue measure with a suitable Radon-Nikodym derivative.

Problem 3:

a) Show that if $t > 0$, $f \in L^2(\mathbb{R}^n)$, and $h(x) = f(x/t)$ for $x \in \mathbb{R}^n$, then $\widehat{h}(\xi) = t^n \widehat{f}(t\xi)$ for a.e. $\xi \in \mathbb{R}^n$ (this and (b) extend statements proved in class).

b) Show that if $f \in L^2(\mathbb{R}^n)$ and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation, then $\widehat{f \circ T} = \widehat{f} \circ T$ a.e.

c) Show that if $f \in L^1(\mathbb{R}^n)$ is a *radial function*, i.e., there exists a function $g\colon [0,\infty) \to \mathbb{C}$ such that $f(x) = g(|x|)$ for all $x \in \mathbb{R}^n$, then $\widehat{f}$ is also a radial function.

Problem 4: We consider the function $f(x) = x$ for $x \in [-\pi, \pi)$, and extend it uniquely to a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$.

a) Compute the Fourier coefficients of f.

b) Use (a) to determine the value $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

c) Use similar ideas to determine the value $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.