Homework 3 (due: Mo, April 30)

Problem 1: For $n \in \mathbb{N}_0$ and $t \in \mathbb{R}$ we consider the Dirichlet kernel

$$D_n(t) = \frac{\sin((n+1)t/2)}{\sin(t/2)}$$

and the Fejér kernel

$$K_n(t) = \frac{1}{n+1}(D_0(t) + \cdots + D_n(t)) = \frac{1}{n+1}\left(\frac{\sin((n+1)t/2)}{\sin(t/2)}\right)^2.$$

a) Show that the kernels K_n have the following properties:
   (i) K_n for $n \in \mathbb{N}_0$ is a non-negative, 2π-periodic, and measurable function on $\mathbb{R}$ with $\frac{1}{2\pi}\int_{[-\pi,\pi]} K_n(t)\,dt = 1$.
   (ii) for each $\delta > 0$ we have $\lim_{n\to\infty} \int_{[-\pi,\pi]\setminus[-\delta,\delta]} K_n(t)\,dt = 0$.

b) Show that a sequence P_n, $n \in \mathbb{N}_0$, of kernels with the properties (i) and (ii) as in (a) forms an *approximate identity* on $\mathbb{T}$ in the following sense: for each $f \in C(\mathbb{T})$ we have

$$(P_n * f)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(x-t)f(t)\,dt \to f(x) \text{ as } n \to \infty$$

uniformly for $x \in \mathbb{R}$.

c) Let $s_n f$ denote the n-th partial sum of the Fourier series of a function $f \in L^1(\mathbb{T})$ and consider

$$\sigma_n f = \frac{1}{n+1}(s_0 f + \cdots + s_n f).$$

Show that if $f \in C(\mathbb{T})$, then

$$\|\sigma_n f - f\|_\infty \to 0 \text{ as } n \to \infty.$$

Problem 2: Let $f(t) = \sum c_n e^{itn}$ be a trigonometric polynomial. Then its *(discrete) Hilbert transform* Hf is defined as

$$Hf(t) = -i\sum_{n\le -1} c_n e^{itn} + i\sum_{n\ge 1} c_n e^{itn}.$$

One can show that the Hilbert transform is bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$. More precisely, there exists a constant $C_p \geq 0$ such that

$$\|Hf\|_p \leq C_p \|f\|_p$$

for each trigonometric polynomial f (this is a rather difficult theorem that you can use without further justification in this problem).

a) Show that for $1 < p < \infty$ the Hilbert transform extends uniquely to a bounded linear operator $H \colon L^p(\mathbb{T}) \to L^p(\mathbb{T})$.

b) For $n \in \mathbb{N}$ let $s_n f$ be the n-th partial sum of the Fourier series of the trigonometric polynomial f. Show that one can represent $s_n f$ as a sum of four terms involving the operator H and two Fourier coefficients of f. Hint: One of the terms is

$$\frac{1}{2i} u_n H(f u_{-n}),$$

where $u_{\pm n}(t) = e^{\pm int}$.

c) Use the previous facts to show that for each $1 < p < \infty$ there exists a constant $C'_p \geq 0$ such that

$$\|s_n f\|_p \leq C'_p \|f\|_p$$

for all $n \in \mathbb{N}_0$ and $f \in L^p(\mathbb{T})$. In other words, the operators $f \mapsto s_n f$ have uniformly bounded operator norms on $L^p(\mathbb{T})$.

d) Use (c) to show that if $f \in L^p(\mathbb{T})$ with $1 < p < \infty$, then the Fourier series of f converges to f in $L^p(\mathbb{T})$, or equivalently,

$$\|s_n f - f\|_p \to 0 \text{ as } n \to \infty.$$

Problem 3: A function $f \colon \mathbb{R}^n \to \mathbb{C}$ is called a *Schwartz function* if it is C^∞-smooth and if all of its partial derivatives $\partial^\alpha f(x)$ tend to 0 as $|x| \to \infty$ faster than any polynomial rate; more precisely, we require that for each multi-index α and each $N \in \mathbb{N}_0$ we have

$$\partial^\alpha f(x) = o((1 + |x|)^{-N}) \text{ as } |x| \to \infty.$$

a) Show that for $f \in C^\infty(\mathbb{R}^n)$ the last condition is equivalent to the following condition: for each multi-index α and each $N \in \mathbb{N}_0$ there exists a constant $C = C(\alpha, N) \geq 0$ such that

$$(1 + |x|)^N |\partial^\alpha f(x)| \leq C \text{ for all } x \in \mathbb{R}^n.$$

b) Show that the function $x \in \mathbb{R}^n \mapsto e^{-|x|^2}$ is a Schwartz function.

c) Show that if f and g are Schwartz functions on $\mathbb{R}^n$, then $(g * f)(x)$ is defined for each $x \in \mathbb{R}^n$ and $f * g$ is also a Schwartz function.

Problem 4: Let $f \in C(\mathbb{T})$ and suppose that $\widehat{f}(n) \geq 0$ for each $n \in \mathbb{Z}$. Show that then $\sum_{n\in\mathbb{Z}} \widehat{f}(n) < \infty$.