Homework 2 (due: Mo, April 23)

Problem 1: Let X be a (complex) Hilbert space.

a) Show that if Y is another Hilbert space and $T\colon X \to Y$ is a bounded linear operator, then there exists a bounded linear operator $T^*\colon Y \to X$ (the *adjoint* of T) such that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for all $x \in X$ and $y \in Y$. Hint: For fixed $y \in Y$ consider the map $x \in X \mapsto \langle T(x), y\rangle \in \mathbb{C}$.

b) Let M be a closed linear subspace of X. We proved that each $x \in X$ can be uniquely represented in the form $x = y + z$ with $y \in M$ and $z \in M^\perp$. Show that $x \mapsto y$ defines a bounded linear operator $P\colon X \to X$ (the *orthogonal projection* of X to M) satisfying $P^2 := P \circ P = P$ and $P^* = P$. What is the operator norm $\|P\|$?

c) Let $P\colon X \to X$ be a bounded linear operator with $P^2 = P$ and $P^* = P$. Show that then there exists a closed linear subspace M of X such that P is the orthogonal projection of X to M as defined in (b).

d) For $N \in \mathbb{N}$ let $\mathcal{P}_N$ denote the space of trigonometric polynomial of degree N, i.e., the set of all functions f of the form

$$f = \sum_{n=-N}^{N} c_n u_n,$$

where $u_n(t) = e^{int}$ and $c_n \in \mathbb{C}$ and for $k = -N, \dots, N$.

Show that $\mathcal{P}_N$ is a closed linear subspace of $L^2(\mathbb{T})$ and that $s_N\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined as

$$s_N(f) = \sum_{n=-N}^{N} \widehat{f}(n) u_n \text{ for } f \in L^2(\mathbb{T})$$

is the orthogonal projection of $L^2(\mathbb{T})$ to $\mathcal{P}_N$.

Problem 2:

a) Let $\{c_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers with

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty.$$

Show that then

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \tag{1}$$

converges uniformly for $t \in \mathbb{R}$ and represents a 2π-periodic continuous function f on $\mathbb{R}$ with $\widehat{f}(n) = c_n$ for $n \in \mathbb{Z}$.

b) Suppose $f\colon \mathbb{R} \to \mathbb{C}$ is a 2π-periodic function in $C^1(\mathbb{R})$. Show that then

$$\widehat{f}(n) = O(1/|n|) \text{ as } |n| \to \infty.$$

c) Let $f \in L^1(\mathbb{T})$. Show that then there exists a C^∞-smooth function $g\colon \mathbb{R} \to \mathbb{C}$ with $f = g$ a.e. if and only if for all $k \in \mathbb{N}$ we have

$$\widehat{f}(n) = O(1/|n|^k) \text{ as } |n| \to \infty.$$

Hint: For one of the directions show that under suitable assumptions the series in (1) can be differentiated term-by-term.

Problem 3: Show that there are functions $f \in C(\mathbb{T})$ such that for the N-th partial sum $s_N(f)$ of its Fourier series we have

$$s_N(f)(0) \not\to f(0) \text{ as } N \to \infty.$$

Hint: Argue by contradiction. Consider the operators $\Lambda_N\colon C(\mathbb{T}) \to \mathbb{C}$, $\Lambda_N(f) = s_N(f)(0)$ for $f \in C(\mathbb{T})$. Show that for their operator norms we have $\|\Lambda_N\| \geq \|D_N\|_1$, where D_N is the Dirichlet kernel.

Problem 4: Let $\alpha \in \mathbb{R}$ be an irrational number.

a) Suppose $f\colon \mathbb{R} \to \mathbb{C}$ is a continuous function satisfying $f(t+1) = f(t)$ for $t \in \mathbb{R}$ (i.e., f has the period 1). Show that then

$$\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) \to \int_0^1 f(t)\,dt \text{ as } N \to \infty.$$

Hint: First show this for certain types of functions f, for which the sum on the left hand side can be computed explicitly.

b) For $x \in \mathbb{R}$ we define $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ and $\operatorname{frac}(x) = x - \lfloor x \rfloor \in [0,1)$. So $\operatorname{frac}(x)$ is the fractional part of x.

Prove *Weyl's Equidistribution Theorem:* for each interval $[a,b] \subseteq [0,1]$ we have

$$\lim_{N\to\infty} \frac{\#\{n \in \{1,\dots,N\} : \operatorname{frac}(n\alpha) \in [a,b]\}}{N} = b - a.$$

Here $\#M$ denotes the number of elements in M.

c) Show that $\{\operatorname{frac}(n\alpha) : n \in \mathbb{N}\}$ is dense in $[0,1]$.