Homework 1 (due: Fr, April 13)

As in 245AB, the homework has to be prepared with the mathematical typesetting system LaTeX.

Problem 1: The purpose of this problem is to fill in some of the details of the proof of Weierstrass's Approximation Theorem outlined in class.

For $n \in \mathbb{N}$ we define the kernel $K_n \colon \mathbb{R} \to [0, \infty)$ by setting

$$K_n(x) = c_n(1 - x^2)^n$$

for $|x| \leq 1$ and $K_n(x) = 0$ for $|x| > 1$. Here $c_n > 0$ is chosen such that $\int K_n(x)\, dx = 1$.

a) Show that for each $\delta > 0$ we have $K_n \to 0$ as $n \to \infty$ uniformly on $\mathbb{R} \setminus (-\delta, \delta)$.

b) Suppose that $f \in C_c(\mathbb{R})$ and $\mathrm{supp}(f) \subseteq [0, 1]$. Define

$$P_n(x) = (K_n * f)(x) = \int K_n(x - u) f(u)\, du$$

for $x \in \mathbb{R}$. Show that for $x \in [0, 1]$ the expression $P_n(x)$ is equal to a polynomial in x.

c) Show that we have uniform convergence $P_n \to f$ as $n \to \infty$ on each compact set $M \subseteq (0, 1)$.

d) Use the previous considerations to prove Weierstrass's Approximation Theorem: if $[a, b] \subseteq \mathbb{R}$ is a compact interval, then the set of polynomials is dense in $C([a, b])$. Hint: First prove this for $[a, b] \subseteq (0, 1)$.

Problem 2: Let $n \in \mathbb{N}$, $n \geq 2$, and $p \in (1, n)$. For a function $f \in L^p(\mathbb{R}^n)$ we consider the *Riesz potential*

$$I(f)(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n-1}}\, dy, \quad x \in \mathbb{R}^n,$$

where dy indicates integration with respect to Lebesgue measure.

a) Fix $x \in \mathbb{R}^n$ and suppose that for some $R > 0$ we have $f = 0$ on $\mathbb{R}^n \backslash B(x, R)$. Show that then

$$I(f)(x) \leq C_1 R (Mf)(x),$$

where $C_1 = C_1(n) > 0$ (i.e., C_1 only depends on n) and Mf denotes the (uncentered) Hardy-Littlewood maximal function of f. Hint: Decompose $B(x, R)$ into dyadic annuli.

b) Fix $x \in \mathbb{R}^n$. Suppose that for some $R > 0$ we have $f = 0$ on $B(x, R)$. Show that then
$$I(f)(x) \le C_2 R^{1-n/p} \|f\|_p,$$
where $C_2 = C_2(p, n) > 0$.

c) Show that
$$\|I(f)\|_{p^*} \le C_3 \|f\|_p,$$
where $p^* = np/(n-p)$ and $C_3 = C_3(p, n) > 0$. Hint: Split the given function f into two functions as suggested by (a) and (b). Optimize R to find a good pointwise estimate for $I(f)(x)$.

Problem 3: Let X be a complex Hilbert space.

a) Show that every orthonormal set $A \subseteq X$ is contained in a maximal orthonormal set $B \subseteq X$.

b) Show if $\{x_n\}_{n\in\mathbb{N}}$ is an orthonormal set in X, then $x_n \to 0$ in the weak topology on X. Hint: This follows from the more general condition in Problem 4, but derive this from Bessel's inequality.

c) Show that if X is separable and infinite-dimensional, then each maximal orthonormal set $A \subseteq X$ is countably infinite. Hint: Use the existence of a countably infinite maximal orthonormal set (=Hilbert space basis) as discussed in class.

d) Show that if X is separable and infinite-dimensional, then there exists a linear isomorphism $T\colon X \to \ell^2$ that preserves the inner product (recall that ℓ^2 is the L^2-space on $\mathbb{N}$ equipped with the counting measure).

Problem 4: Let X be a complex separable Hilbert space with a Hilbert space basis $\{x_n\}_{n\in\mathbb{N}}$.

a) Show that the infinite series $\displaystyle\sum_{n=1}^{\infty} \alpha_n x_n$ with coefficients $\alpha_n \in \mathbb{C}$, $n \in \mathbb{N}$, converges in X if and only if $\displaystyle\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$.

b) Show that for a sequence $\{y_n\}_{n\in\mathbb{N}}$ in X we have $y_n \to y \in X$ in the weak topology on X if and only if the following conditions are true:
   (i) There exists a constant $C \ge 0$ such that $\|y_n\| \le C$ for all $n \in \mathbb{N}$.
   (ii) We have $\langle y_n, x_k\rangle \to \langle y, x_k\rangle$ as $n \to \infty$ for each $k \in \mathbb{N}$.

c) Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in X. Show that then there exists $y \in Y$ such that $y_n \xrightarrow{w} y$ if and only the following conditions are true:
   (i) There exists a constant $C \ge 0$ such that $\|y_n\| \le C$ for all $n \in \mathbb{N}$.
   (ii) The sequence $\{\langle y_n, x\rangle\}_{n\in\mathbb{N}}$ converges for each $x \in X$.