Math 245C, Real Analysis

Spring 2018

Midterm

Name:

There are five problems with a total of 50 points. The Midterm is due on Friday, May 25, before class.

Problem 1: In the following we assume that $f \in C(\mathbb{R})$ and that $|f(x)| = O(\frac{1}{|x|^{1+\epsilon}})$ as $|x| \to \infty$ for some $\epsilon > 0$.

(a) Show that $f \in L^1(\mathbb{R})$ and that the series

$$F(x) = \sum_{n \in \mathbb{Z}} f(x+n) = \lim_{N \to \infty} \sum_{n=-N}^{N} f(x+n)$$

converges uniformly in x on each compact subset of $\mathbb{R}$. (4pts)

Note that by (a) the Fourier transform $\widehat{f}$ of f is defined. We assume in addition that $|\widehat{f}(x)| = O(\frac{1}{|x|^{1+\epsilon'}})$ as $|x| \to \infty$ for some $\epsilon' > 0$.

(b) Define $G(u) = F(\frac{u}{2\pi})$ for $u \in \mathbb{R}$. Show that then G is a 2π-periodic function in $L^1(\mathbb{T})$ and express the Fourier coefficients $\widehat{G}(n)$ of G in terms of the Fourier transform $\widehat{f}$ of f. (3pts)

(c) Show that

$$\sum_{n \in \mathbb{Z}} f(x+n) = \sqrt{2\pi} \sum_{n \in \mathbb{Z}} \widehat{f}(2\pi n) e^{2\pi i n x}$$

for all $x \in \mathbb{R}$. This is known as *Poisson's summation formula.* (4pts)

Problem 2: Use Poisson's summation formula as derived in Problem 1 to show the following identities:

(a) $\displaystyle\sum_{n\in\mathbb{Z}} \frac{a}{a^2+n^2} = \pi \frac{e^{2a\pi}+1}{e^{2a\pi}-1}$ for all $a > 0$. (5pts)

(b) $\displaystyle\sum_{n\in\mathbb{Z}} e^{-\pi n^2/x} = \sqrt{x}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$ for all $x > 0$. (5pts)

Problem 3: The purpose of this problem is to describe all distributions T on $\mathbb{R}$ with compact support $\operatorname{supp}(T)$. For simplicity we assume that $\operatorname{supp}(T) \subseteq (0,1)$.

(a) Show that there exist constants $n \in \mathbb{N}$ and $C > 0$ such that

$$|T(\phi)| \leq C \int_0^1 |\phi^{(n)}(x)|\, dx$$

for all $\phi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\phi) \subseteq [0,1]$. Hint: Use a related inequality proved in class. (4pts)

(b) If $\phi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\phi) \subseteq [0,1]$, then we set $L(\phi^{(n)}) := T(\phi)$. Show that L is well-defined and can be extended to a bounded linear functional on $L^1([0,1])$. (3pts)

(c) Show that there exists $g \in L^1_{loc}(\mathbb{R})$ such that $T = g^{(n)}$. Here $g^{(n)}$ is the n-th distributional derivative of g. (3pts)

Problem 4: (a) The Laplacian $\Delta\phi$ of a sufficiently smooth function $\phi = \phi(x, y)$ on $\mathbb{R}^2$ is defined as

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2},$$

where x and y are the standard coordinate functions on $\mathbb{R}^2$.

If we introduce polar coordinates r and α on $\mathbb{R}^2$ by setting $x = r\cos\alpha$ and $y = r\sin\alpha$, then we can consider ϕ as a function of r and α. Express $\Delta\phi$ in terms of partial derivatives with respect to r and α. (4 pts)

(b) Define $f(u) = \log|u|$ for $u = (x, y) \in \mathbb{R}^2$ and consider this as a distribution on $\mathbb{R}^2$. Show that the distribution Δf is a (non-trivial) measure on $\mathbb{R}^2$. (6 pts)

Problem 5: We denote by $\widehat{f}$ the Fourier-Plancherel transform of a function $f \in L^2(\mathbb{R}^n)$.

(a) Show that

$$\int f\widehat{g} = \int \widehat{f} g$$

for all $f, g \in L^2(\mathbb{R}^n)$. (2 pts)

(b) Let $f \in W^{1,2}(\mathbb{R}^n)$, and $\partial_k f$ be the weak k-th partial derivative of f, where $k \in \{1, \dots, n\}$. Show that

$$\int \partial_k f s = -\int f \partial_k s$$

for all Schwartz functions $s \in \mathcal{S}(\mathbb{R}^n)$. (3 pts)

(c) Show that if $f \in W^{1,2}(\mathbb{R}^n)$, then $\widehat{\partial_k f}(\xi) = i\xi_k \widehat{f}(\xi)$ for all $k \in \{1, \dots, n\}$ and almost every $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Hint: Use (a) and (b). (2 pts)

(d) Show that if $f \in L^2(\mathbb{R}^n)$, then $f \in W^{1,2}(\mathbb{R}^n)$ if and only if

$$\int (1 + |\xi|^2)|\widehat{f}(\xi)|^2 \, d\xi < \infty.$$

(3 pts)