Math 245C, Real Analysis

Spring 2018

Final Exam

Name:

There are five problems with a total of 50 points. The exam is due by Tuesday, June 12, 12pm.

Problem 1: We consider the Sobolev space $W^{1,p}(\mathbb{R}^n)$ for fixed $1 \leq p < \infty$. As usual,

$$\|f\|_{1,p} = \left(\int |f|^p + \sum_{k=1}^{n} \int |\partial_k f|^p \right)^{1/p}$$

denotes the Sobolev norm of a function $f \in W^{1,p}(\mathbb{R}^n)$. The purpose of this problem is to show that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$.

(a) Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a mollifier, and

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi(x/\epsilon) \text{ for } x \in \mathbb{R}^n,\ \epsilon > 0.$$

Show that if $f \in W^{1,p}(\mathbb{R}^n)$ and we define $g_k = \varphi_{1/k} * f$ for $k \in \mathbb{N}$, then $g_k(x) \to f(x)$ for a.e. $x \in \mathbb{R}^n$ and $\|g_k - f\|_{1,p} \to 0$ as $k \to \infty$. (5pts)

(b) Show that if $f \in W^{1,p}(\mathbb{R}^n)$, then there exist functions $f_k \in C_c^\infty(\mathbb{R}^n)$ for $k \in \mathbb{N}$ such that $f_k(x) \to f(x)$ for a.e. $x \in \mathbb{R}^n$ and $\|f_k - f\|_{1,p} \to 0$ as $k \to \infty$. (5pts)

Problem 2: We fix $p \in (n, \infty)$. If $f \in L^1_{loc}(\mathbb{R}^n)$ and $B \subseteq R^n$ is a ball, then we denote by

$$f_B := \frac{1}{\lambda_n(B)} \int f \, d\lambda_n$$

the average of f over B.

(a) Let $f \in C^1(\mathbb{R}^n)$, $B \subseteq \mathbb{R}^n$ be a ball with radius $r > 0$, and $x \in B$. Show that then

$$|f(x) - f_B| \leq C r^{1-n/p} \left(\int_B |\nabla f|^p \right)^{1/p},$$

where $C = C(p, n) > 0$ is a constant only depending on p and n. Hint: Express $|f(x) - f(y)|$ for $y \in B$ in terms of ∇f and integrate over y. (4pts)

(b) Show that if $f \in C^1(\mathbb{R}^n)$ and $\int |\nabla f|^p < \infty$, then f is α-Hölder continuous with $\alpha = 1 - n/p$. More precisely, show that there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for all $x, y \in \mathbb{R}^n$. (2pts)

(c) Prove the following version of *Sobolev's embedding theorem* (for the "supercritical" exponent $p \in (n, \infty)$): if $f \in W^{1,p}(\mathbb{R}^n)$, then f has an α-Hölder continuous representative with $\alpha = 1 - n/p$ in its Sobolev class. More precisely, there exists an α-Hölder continuous function $g \colon \mathbb{R}^n \to \mathbb{C}$ such that $f = g$ a.e. on $\mathbb{R}^n$. Hint: Use Problem 1 and show that f is α-Hölder continuous on a dense subset of $\mathbb{R}^n$. (4pts)

Problem 3: The purpose of this problem is to determine all distributions T on $\mathbb{R}^n$ with $\operatorname{supp}(T) = \{0\}$. In the following, T will be such a distribution. By a theorem proved in class, we know that there exists $N \in \mathbb{N}_0$ and a constant $C > 0$ such that

$$|T(\varphi)| \leq C\|\varphi\|_{N,\infty}$$

for each $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\varphi) \subseteq B(0,1)$, where

$$\|\varphi\|_{N,\infty} = \max\{|\partial^\alpha \varphi(x)| : x \in \mathbb{R}^n,\ |\alpha| \leq N\}.$$

(a) Suppose $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\partial^\alpha \varphi(0) = 0$ for all multi-indices α with $|\alpha| \leq N$. Let $\epsilon > 0$ be arbitrary. Then we can find a ball $B \subseteq \mathbb{R}^n$ centered at 0 such that

$$|\partial^\alpha \varphi(x)| \leq \epsilon$$

for $x \in B$ and all α with $|\alpha| = N$ (why?). Show that then

$$|\partial^\alpha \varphi(x)| \leq n^{N-|\alpha|}|x|^{N-|\alpha|}\epsilon$$

for $x \in B$ and $|\alpha| \leq N$. (2pts)

(b) We can find a function $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\psi) \subseteq B(0,1)$ and $\psi = 1$ in a neighborhood of 0. Define $\psi_r(x) = \psi(x/r)$ for $r > 0$ and $x \in \mathbb{R}^n$. Show that if $r > 0$ is small enough, then

$$\|\psi_r \varphi\|_{N,\infty} \leq \epsilon C \|\psi\|_{N,\infty},$$

where φ is as in (a) and $C = C(n,N) > 0$ is a constant only depending on n and N. (2pts)

(c) Show that if $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\partial^\alpha \varphi(0) = 0$ for all α with $|\alpha| \leq N$, then $T(\varphi) = 0$. (2pts)

(d) Show that there are constants $c_\alpha \in \mathbb{C}$ such that

$$T = \sum_{|\alpha| \leq N} c_\alpha \partial^\alpha \delta,$$

where δ denotes the Dirac delta measure at 0. Hint: If a linear functional L on a vector space vanishes on the intersection of the kernels of the linear functionals $L_1, \ldots, L_k$, then L can be represented as a linear combination of $L_1, \ldots, L_k$ (if you want to use this fact, you have to prove it first!). (4pts)

Problem 4: Let T be a tempered distribution on $\mathbb{R}^n$ and suppose that $\Delta T = 0$, where

$$\Delta = \frac{\partial^2}{dx_1^2} + \cdots + \frac{\partial^2}{dx_n^2}$$

is the Laplace operator written with the standard coordinates $x_1, \ldots, x_n$ on $\mathbb{R}^n$. Show that then T is a polynomial P (in $x_1, \ldots, x_n$) satisfying $\Delta P = 0$. Hint: Use Problem 3. (10pts)

Problem 5: We consider the standard 1/3-Cantor set $C \subseteq \mathbb{R}$ consisting of all points $x \in \mathbb{R}$ that can be written in the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$$

with $a_i \in \{0, 2\}$ for $i \in \mathbb{N}$. For $n \in \mathbb{N}$ we consider the set $D_n \subseteq C$ consisting of all $x \in \mathbb{R}$ of the form

$$x = \sum_{i=1}^{n} \frac{a_i}{3^i}$$

with $a_i \in \{0, 2\}$ for $1 \leq i \leq n$.

(a) For $n \in \mathbb{N}$ and $f \in C_c(\mathbb{R})$ we define $I_n(f) = \dfrac{1}{2^n} \displaystyle\sum_{x \in D_n} f(x)$. Show that then the limit $I(f) := \lim\limits_{n \to \infty} I_n(f)$ exists and that there exists a positive Borel measure μ on $\mathbb{R}$ such that $I(f) = \displaystyle\int f \, d\mu$ for each $f \in C_c(\mathbb{R})$. (2pts)

(b) Show that μ is a probability measure concentrated on C, i.e., $\mu(C) = 1$ and $\mu(\mathbb{R} \backslash C) = 0$, and that $\mu([x, x + 1/3^n]) = 1/2^n$ for each $n \in \mathbb{N}$ and $x \in D_n$. (2pts)

(c) Show that if $\alpha > \log 2 / \log 3$, then $H^\alpha(C) = 0$ for the α-Hausdorff measure of C. Hint: Use an efficient cover of C. (2pts)

(d) Show that if $\alpha = \log 2 / \log 3$, then $H^\alpha(C) > 0$ and conclude that $\dim_H C = \log 2 / \log 3$ for the Hausdorff dimension of C. Hint: $\mu(B(x, r)) \lesssim r^\alpha$. (4pts)