Exercises

Problem 1: a) Let $[a,b] \subseteq \mathbb{R}$ be a compact interval and $f, g\colon [a,b] \to \mathbb{C}$ be absolutely continuous functions.

(a) Show that $h = fg$ is also absolutely continuous on $[a,b]$.

(b) Prove the following integration-by-parts formula:

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$

Justify the existence of all derivatives and integrals involved!

Problem 2: Show that if $f \in L^1_{loc}(\mathbb{R})$, then there exists $g \in L^1_{loc}(\mathbb{R})$ such that $g' = f$ in the sense of distributions.

Problem 3: Show that the following statement false in general: if $f, g \in L^1_{loc}(\mathbb{R})$ and $f' = g$ in the distributional sense, then $f'(x)$ exists in the classical sense and $f'(x) = g(x)$ for almost every $x \in \mathbb{R}$.

Problem 4: Let $s_k \in \mathcal{S}(\mathbb{R}^n)$ for $k \in \mathbb{N} \cup \{\infty\}$ and suppose that $s_k \to s_\infty$ as $k \to \infty$ in the topology of $\mathcal{S}(\mathbb{R}^n)$. Show that then for each multi-index α and for each $1 \le p \le \infty$ we have $\|\partial^\alpha s_k - \partial^\alpha s_\infty\|_p \to 0$ as $k \to \infty$ (here $\|\cdot\|_p$ denotes the L^p-norm).

Problem 5: Let $f \in L^1_{loc}(\mathbb{R}^n)$ and suppose that $\partial_k f = 0$ in the distributional sense for all $k = 1, \dots, n$. Show that then there exists a constant $c \in \mathbb{C}$ such that $f = c$ a.e. on $\mathbb{R}$. Hint: Consider a mollification $\varphi_\epsilon * f$.

Problem 6: (a) Let $a \in \mathbb{R}^n$, $R > 0$, and consider the sphere

$$\Sigma(a,R) = \{x \in \mathbb{R}^n : |x - a| = R\}$$

of radius R centered at a. Then $\Sigma(a,R)$ carries a natural Borel measure σ uniquely determined by the relation

$$\sigma(B) = n\lambda_n(\widetilde{B})$$

for each Borel set $B \subseteq \Sigma(a,R)$. Here $\widetilde{B}$ is the cone with tip a and base B, i.e,

$$\widetilde{B} = \{a + t(b-a) : b \in B,\, t \in (0,1)\}.$$

Note that this is very similar to how we defined spherical measure on the unit sphere $\Sigma(0,1)$.

Let $f\colon \Sigma(a,R) \to [0,\infty]$ be a Borel function.

(a) Show that

$$\int_{\Sigma(a,R)} f\,d\sigma = R^{n-1}\int_{\Sigma(0,1)} f(a+R\xi)\,d\sigma(\xi).$$

(b) Show that

$$\int_{\Sigma(a,R)} f\,d\sigma = R^{n-1}\int_{B_{n-1}} \frac{1}{y_n}[f(a+R(y,y_n)) - f(a+R(y,-y_n))]\,d\lambda_{n-1}(y).$$

Here B_{n-1} is the open unit ball in $\mathbb{R}^{n-1}$ and we set $y_n = \sqrt{1-|y|^2}$ for $y \in B_{n-1}$.

Problem 7: We use the notation from Problem 6. We denote by

$$n(x) = \frac{1}{|x-a|}(x-a)$$

the unit normal vector to the hypersurface $\Sigma(a,R)$ at a point $x \in \Sigma(a,R) = \partial B(a,R)$ pointing "outward". Let $v = (v_1,\dots,v_n)$ be a C^1-smooth vector field (i.e., an $\mathbb{R}^n$-valued function) defined in an open neighborhood of $\overline{B}(a,R)$.

(a) Show that then

$$\int_{B(a,R)} \operatorname{div} v\,d\lambda_n = \int_{\Sigma(a,R)} v\cdot n\,d\sigma.$$

Here

$$\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n}$$

with the standard coordinate functions $x_1,\dots,x_n$. This is a special case of *Gauss's Theorem*. Hint: First consider vector fields of the form $v = (0,\dots,0,v_n)$ and use Problem 6.

(b) Prove the following more general version of Gauss's Theorem. Consider the an open set $\Omega \subseteq \mathbb{R}^n$ of the form $\Omega = B_0\backslash(B_1\cup\cdots\cup B_k)$, where B_0 is an open ball in $\mathbb{R}^n$ and $B_1,\dots,B_k$ are pairwise disjoint closed balls in $\mathbb{R}^n$ contained in B_0. Suppose v is a C^1-smooth vector field defined in an open neighborhood of $\overline{\Omega}$. Then

$$\int_{\Omega} \operatorname{div} v\,d\lambda_n = \int_{\partial\Omega} v\cdot n\,d\sigma.$$

Here n again denotes the "outward" normal unit vector defined on

$$\partial\Omega = \partial B_0 \cup\cdots\cup \partial B_k.$$

So on ∂B_0 it is equal to the surface normal defined in (a), but on $\partial B_1,\dots,\partial B_k$ it differs by a sign.

Problem 8: (a) Let $\Omega \subseteq \mathbb{R}^n$ be an open set as in Problem 7, and f,g be C^2-smooth functions defined in an open neighborhood of $\overline{\Omega}$. Use Gauss's Theorem to prove the *Gauss-Green formula*:

$$\int_{\Omega} (f\Delta g - g\Delta f)\, d\lambda_n = \int_{\partial\Omega} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) d\sigma.$$

Here

$$\frac{\partial h}{\partial n} = \nabla h \cdot n$$

denotes the derivative of the function h in the direction of the outward normal n at a point $x \in \partial\Omega$.

(b) Use the Gauss-Green formula to give an alternative proof of the fact that on $\mathbb{R}^2$ we have

$$\Delta_x \log|x| = 2\pi\delta_0$$

in the distributional sense.

Problem 9: Find a *fundamental solution* of the Laplace equation in all dimensions $n \in \mathbb{N}$, i.e., a distribution T on $\mathbb{R}^n$ such that

$$\Delta T = \delta_0.$$