Homework 9 (due: Fr, Mar. 16)

Problem 1: Let $1 \leq p < \infty$.

a) Let $f \in L^p(\mathbb{R}^n)$. For $y \in \mathbb{R}^n$ we define $\tau_y f \colon \mathbb{R}^n \to \mathbb{C}$ as
$$\tau_y f(x) = f(x+y), \quad x \in \mathbb{R}^n.$$
Show that $\tau_y f \in L^p(\mathbb{R}^n)$ and that the operator τ_y has the following continuity property:
$$\lim_{z \to y} \|\tau_z f - \tau_y f\|_p = 0.$$
Hint: Approximate f by a function in $C_c(\mathbb{R}^n)$.

b) Let $C_c^\infty(\mathbb{R}^n)$ be the space of all C^∞-smooth functions on $\mathbb{R}^n$ with compact support, and $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a mollifier (see HW 7, Prob. 4). For $t > 0$ we define
$$\varphi_t(x) = \frac{1}{t^n}\varphi(x/t), \quad x \in \mathbb{R}^n.$$
Show that if $f \in L^p(\mathbb{R}^n)$, then $f * \varphi_t \in L^p(\mathbb{R}^n)$ and
$$f * \varphi_t \to f \quad \text{in } L^p(\mathbb{R}^n) \text{ as } t \to 0^+.$$

c) Show that the space $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Hint: HW 8, Prob. 1.

Problem 2: Let X be a compact topological space. We assume that there exists a countable family $\{f_n \colon X \to \mathbb{C} : n \in \mathbb{N}\}$ of continuous complex-valued functions on X that *separates points* in the following sense: for all $x, y \in X$ with $x \neq y$ there exists $n \in \mathbb{N}$ such that $f_n(x) \neq f_n(y)$. The purpose of this problem is to show that then X is *metrizable*, i.e., there exists a metric d on X that induces the given topology on X.

a) We may assume that $|f_n| \leq 1$ for each $n \in \mathbb{N}$ (why?). Show that if we define
$$d(x,y) = \sum_{n=1}^\infty \frac{1}{2^n}|f_n(x) - f_n(y)|$$
for $x, y \in X$, then d is a metric on X.

b) Show that if $g \colon Y \to Z$ is a continuous bijection between a compact space Y and a Hausdorff space Z, then g is a homeomorphism.

c) Use (a) and (b) to show that X is metrizable.

d) Let V be a separable Banach space and $B = \{f \in V^* : \|f\| \leq 1\}$ be the closed unit ball in the dual space V^* of V. Show that if we equip B with the weak-$*$ topology, then B is metrizable.

Problem 3: Let (X, d) be a compact metric space.

a) Suppose that for each $n \in \mathbb{N}$ we have a finite set $F_n \subseteq X$ such that the open balls $B(y, 1/n)$, $y \in F_n$, form an open cover $\mathcal{U}_n$ of X. Let g_y, $y \in F_n$, be a the functions in a partition of unity subordinate to $\mathcal{U}_n$. If $f \in C(X)$ is a continuous complex-valued function on X, we define

$$f_n = \sum_{y \in F_n} f(y) g_y.$$

Show that then $f_n \to f$ uniformly on X.

b) Show that the Banach space $C(X)$ of all continuous complex-valued functions on X is separable.

c) Show that if $\{\mu_n\}$ is a sequence of Borel probability measures on X, then the sequence *subconverges* (i.e., some subsequence converges) to a Borel probability measure μ on X in the weak-$*$ topology on the space $\mathcal{M}(X)$ of all complex Borel measures on X.

Problem 4: (Analysis Qualifying Exam, Fall 2014) Let $\{f_n\}$ be a bounded sequence in $L^2(\mathbb{R})$ and suppose that $f_n(x) \to 0$ as $n \to \infty$ for almost every $x \in \mathbb{R}$. Show that then $f_n \to 0$ in the weak topology on $L^2(\mathbb{R})$.