Homework 8 (due: Fr, Mar. 9)

Problem 1: Let f and g be a functions on $\mathbb{R}^n$. Suppose f is locally integrable and g is C^∞-smooth with compact support. Show that then the convolution

$$h(x) = (f * g)(x) = \int f(x-y)g(y)\, d\lambda_n(y)$$

is defined for each $x \in \mathbb{R}^n$. Moreover, h is C^∞-smooth and if $\mathscr{D}$ denotes any partial derivative of any order, then $f * \mathscr{D}g$ is everywhere defined and

$$\mathscr{D}h = f * \mathscr{D}g.$$

Problem 2:

a) Let $\mathcal{O}$ be the family of all sets $U \subseteq \mathbb{R}$ such that $U = \emptyset$ or $\mathbb{R} \setminus U$ is a countable set. Show that $\mathcal{O}$ is a topology on $\mathbb{R}$.

b) Find a simple necessary and sufficient condition for a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ to converge to a point $x \in \mathbb{R}$ in the topological space $(\mathbb{R}, \mathcal{O})$.

c) Show that in $(\mathbb{R}, \mathcal{O})$ not every set is closed, but every set $M \subseteq \mathbb{R}$ has the following property: if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in M and $x_n \to x$, then $x \in M$.

Problem 3:

a) Show that in a Hausdorff space X every convergent net has a unique limit; in other words, if $\{x_\alpha\}_{\alpha\in A}$ is a net in X with $x_\alpha \to x$ and $x_\alpha \to y$, then $x = y$.

b) Show that for nets in $\mathbb{C}$ similar computational rules hold as for sequences: if $\{z_\alpha\}_{\alpha\in A}$ and $\{w_\alpha\}_{\alpha\in A}$ are nets in $\mathbb{C}$ (indexed by the same directed set A) and $z_\alpha \to z$ and $w_\alpha \to w$, then $z_\alpha + w_\alpha \to z + w$ and $z_\alpha w_\alpha \to zw$.

c) A topology on a space is uniquely determined by its convergent nets and their limits: Let X be a set with two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$. Suppose that with respect to these topologies convergent nets are the same with identical limits; more precisely, assume that for every net $\{x_\alpha\}_{\alpha\in A}$ in X and every point $x \in X$ we have $x_\alpha \to x$ in $(X, \mathcal{O}_1)$ if and only if $x_\alpha \to x$ in $(X, \mathcal{O}_2)$. Show that then $\mathcal{O}_1 = \mathcal{O}_2$.

Problem 4: Let X be a Banach space equipped with its weak topology.

a) Show that X is a Hausdorff space.

b) Prove that vector addition is continuous in the weak topology: Consider the map $A\colon X \times X \to X$ given by $A((x, y)) = x + y$ for $(x, y) \in X \times X$. Show that A is continuous. Here $X \times X$ carries the product topology induced by the weak topology on each factor.