Homework 7 (due: Fr, Mar. 2)

Problem 1:

a) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $1 < p < \infty$, and q be the conjugate exponent of p. Show that if $f\colon (X, \mathcal{A}) \to [0, \infty]$ is a measurable function, then

$$\|f\|_p = \sup\left\{ \int fg\, d\mu : g\colon X \to [0,\infty] \text{ measurable, } \|g\|_q \leq 1 \right\} \in [0, \infty].$$

Hint: This does not directly follow from L^p-L^q-duality, because we allow $\|f\|_p = \infty$ here and pair f only with *non-negative* functions g.

b) Prove *Minkowski's inequality for integrals*: Let $1 \leq p < \infty$. Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces and

$$h\colon (X \times Y, \mathcal{A} \otimes \mathcal{B}) \to [0, \infty]$$

is measurable. Then

$$\left(\int \left(\int h(x,y)\, d\nu(y) \right)^p d\mu(x) \right)^{1/p} \leq \int \left(\int h(x,y)^p\, d\mu(x) \right)^{1/p} d\nu(y).$$

c) Show how derive the usual Minkowski inequality $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ from the integral version in (b).

Problem 2: Show that the L^1-boundedness of the Hardy-Littlewood maximal function fails: If $f \in L^1(\mathbb{R}^n)$ is an arbitrary integrable function on $\mathbb{R}^n$ with $f \neq 0$, then $Mf \notin L^1(\mathbb{R}^n)$. Hint: Find a lower bound for Mf that implies $\int Mf = +\infty$.

Problem 3: Let $f\colon [-1, 1] \to \mathbb{C}$ be a continuous function.

a) Show that if f is absolutely continuous on $[-1, 0]$ and on $[0, 1]$, then f is absolutely continuous on $[-1, 1]$.

b) Show (by finding a suitable counterexample) that if f is absolutely continuous on each closed subinterval of $[-1, 1] \setminus \{0\}$, then f may fail to be absolutely continuous.

c) Let $E \subseteq [-1, 1]$ be a closed set with $\lambda_1(E) = 0$. Show that if f is absolutely continuous on each closed subinterval of $[-1, 1] \setminus E$, then $f'(x)$ exists for almost every $x \in [-1, 1]$.

d) Assume in addition to the conditions in (c) that E is countable and that the almost everywhere defined derivative f' of f is integrable on $[-1, 1]$. Show that then f is absolutely continuous on $[-1, 1]$.

e) Show that the conclusion in (d) fails if we do not assume that E is countable.

Problem 4: A function φ on $\mathbb{R}^n$ is called a *mollifier* if it is a non-negative function in $C_c^\infty(\mathbb{R}^n)$ (the space of all C^∞-smooth functions on $\mathbb{R}^n$ with compact support), and $\int \varphi \, d\lambda_n = 1$.

a) Show that mollifiers on $\mathbb{R}^n$ exist for each $n \in \mathbb{N}$ (Hint: HW 3, Prob. 3).

b) Fix a mollifier φ on $\mathbb{R}^n$. If $t > 0$ we define

$$\varphi_t(x) = \frac{1}{t^n}\varphi(x/t), \quad x \in \mathbb{R}^n.$$

If $f \in L^1_{loc}(\mathbb{R}^n)$, we consider the *$t$-mollification* of f defined as

$$(f * \varphi_t)(x) = \int f(x-y)\varphi_t(y)\, d\lambda_n(y), \quad x \in \mathbb{R}^n.$$

Show that $(f * \varphi_t)(x) \to f(x)$ as $t \to 0$ for almost every $x \in \mathbb{R}^n$.

c) Let $g, h \in L^1_{loc}(\mathbb{R}^n)$ and suppose that

$$\int fg \, d\lambda_n = \int fh \, d\lambda_n$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. Show that then $g = h$ almost everywhere on $\mathbb{R}^n$.